

# A HOMOTOPY THEORY OF ADDITIVE CATEGORIES WITH SUSPENSIONS

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**ABSTRACT.** We define partial one-sided triangulated categories by weakening the axioms of one-sided triangulated categories. We show that additive categories, exact categories, one-sided triangulated categories, complete cotorsion pairs in exact categories, torsion pairs and mutation pairs in triangulated categories extend to partial one-sided triangulated categories. We prove that partial one-sided triangulated categories yield one-sided triangulated categories by passing to stable categories. This unifies and generalizes many constructions of stable one-sided triangulated categories, and even stable exact categories. Moreover, it also allows us to model Iyama-Yoshino subfactor triangulated categories via Quillen closed model structures.

## CONTENTS

1. Introduction	1
2. Partial one-sided triangulated categories	3
3. A homotopy theory of partial left triangulated categories	9
4. Examples of partial one-sided triangulated categories	15
5. Stable exact categories	23
6. Stable triangulated categories	27
References	31

## 1. INTRODUCTION

The stable categories of additive categories play central roles in the study of abelian categories and triangulated categories. They provide a bridge between

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abelian categories and triangulated categories. For example, the stable categories of Frobenius categories are triangulated categories [19, 17], in the converse direction, the stable categories of certain triangulated categories are abelian categories [27, 29]. Recently, Iyama and Yoshino observed that certain stable categories of triangulated categories are again triangulated categories (Iyama-Yoshino subfactor triangulated categories) [24]. Meanwhile, Kussin, Lenzen and Meltz observed that the stable categories of some exact categories are again exact categories [30].

The aim of this paper is to develop an axiomatization of homotopy theory of additive categories with suspensions [19] based on the notion of a partial one-sided triangulated category. This homotopy theory provides a unified framework to formulate the various structures mentioned above on the stable categories arising from exact categories and triangulated categories.

In Section 2, we introduce the new notion of partial one-sided triangulated categories. Roughly speaking, a partial left triangulated category is an additive category  $\mathcal{A}$  endowed with an endofunctor  $\Omega$ , a class  $\mathcal{L}(\mathcal{C}, \mathcal{X})$  of special left triangles induced by two additive subcategories  $\mathcal{C}$  and  $\mathcal{X}$  of  $\mathcal{A}$ , satisfying the similar axioms of a left triangulated category [7] except the rotation axiom. We show that additive categories, exact categories and one-sided triangulated categories can always be viewed as partial one-sided triangulated categories.

In Section 3, we show that if  $(\mathcal{A}, \Omega, \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is a partial left triangulated category, then the stable category  $\mathcal{C}/\mathcal{X}$  has a left triangulated structure induced by  $\mathcal{L}(\mathcal{C}, \mathcal{X})$ . Following Quillen, the stable category  $\mathcal{C}/\mathcal{X}$  together with the left triangulated structure is called the *homotopy theory* of the partial left triangulated category. This homotopy theory unifies and generalizes [7, Theorem 2.12]; [5, Theorem 7.1]; [35, Theorem 3.9] and [32, Theorem 3.7].

In Section 4, we show how to construct partial one-sided triangulated categories inside additive categories, exact categories and one-sided triangulated categories. In particular, we see that partial one-sided triangulated categories arise naturally from complete cotorsion pairs in exact categories, and torsion pairs and mutation pairs in the sense of [24] in triangulated categories.

In Section 5, we use the homotopy theory of partial one-sided triangulated categories to construct stable abelian categories from triangulated categories and stable exact categories from exact categories. The former generalizes [29, Theorem 3.3] and the later improves and generalizes [30, Theorem A (2)].

In Section 6, we introduce the notion of a partial triangulated category and show that the homotopy theory of partial triangulated categories unifies the constructions of stable triangulated categories in [17, 24, 38]. As an application, we show that Iyama-Yoshino subfactor triangulated categories admit Quillen closed model structures as constructed in [4].

Throughout this paper, unless otherwise stated, that all subcategories of additive categories considered are full, closed under isomorphisms, all functors between additive categories are assumed to be additive.

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## 2. PARTIAL ONE-SIDED TRIANGULATED CATEGORIES

In this section we introduce the notion of partial one-sided triangulated categories and give some baby examples of partial one-sided triangulated categories.

**2.1. Stable categories of additive categories.** Let  $\mathcal{C}$  be an additive category and  $\mathcal{X}$  an additive subcategory of  $\mathcal{C}$ . Given morphisms  $c, c': C \rightarrow D$  in  $\mathcal{C}$ , we say that  $c$  is *stably equivalent* to  $c'$ , written  $c \sim c'$ , if  $c - c'$  factors through  $\mathcal{X}$  (that is, there exists some object  $X \in \mathcal{X}$  such that there are two morphisms  $u: X \rightarrow D$  and  $v: C \rightarrow X$  satisfying  $c - c' = u \circ v$ ). We use  $\underline{c}$  to denote the stable equivalence class of  $c$ . It is well known that stable equivalence is an equivalence relation which is compatible with composition. That is, if  $c \sim c'$ , then  $c \circ k \sim c' \circ k$  and  $h \circ c \sim h \circ c'$  whenever the compositions make sense. The *stable category*  $\mathcal{C}/\mathcal{X}$  is the category whose objects are the same with  $\mathcal{C}$ , and whose morphisms are the stable equivalence classes of  $\mathcal{C}$ . Recall that the stable category  $\mathcal{C}/\mathcal{X}$  is an additive category. For an object  $A$  in  $\mathcal{C}$ , we will denote it by  $\underline{A}$  when it is in the stable category  $\mathcal{C}/\mathcal{X}$ .

**2.2. Left triangulated categories.** We recall the definition of a left triangulated category.

**Definition 2.1.** ([7, Definition 2.2]) Let  $\mathcal{T}$  be an additive category and  $\Omega$  an additive covariant endofunctor on  $\mathcal{T}$ . Let  $\triangle$  be a class of left triangles of the form

$\Omega(A) \rightarrow C \rightarrow B \rightarrow A$ . The triple  $(\mathcal{T}, \Omega, \Delta)$  is called a *left triangulated category* if  $\Delta$  is closed under isomorphisms and the following four axioms hold:

(LT1) For any morphism  $a: B \rightarrow A$  there is a left triangle in  $\Delta$  of the form  $\Omega(A) \rightarrow C \rightarrow B \xrightarrow{a} A$ . For any object  $A \in \mathcal{C}$ , the left triangle  $0 \rightarrow A \xrightarrow{1_A} A \rightarrow 0$  is in  $\Delta$ .

(LT2) For any left triangle  $\Omega(A) \xrightarrow{c} C \xrightarrow{b} B \xrightarrow{a} A$  in  $\Delta$ , the left triangle  $\Omega(B) \xrightarrow{-\Omega(a)} \Omega(C) \xrightarrow{b} B$  is also in  $\Delta$ .

(LT3) For every diagram of the form

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{c} & C & \xrightarrow{b} & B & \xrightarrow{a} & A \\ \Omega(\alpha) \downarrow & & \gamma \downarrow & & \beta \downarrow & & \downarrow \alpha \\ \Omega(A') & \xrightarrow{c'} & C' & \xrightarrow{b'} & B' & \xrightarrow{a'} & A' \end{array}$$

whose rows are in  $\Delta$  with  $\alpha \circ a = a' \circ \beta$ , there exists  $\gamma: C \rightarrow C'$  such that  $\beta \circ b = b' \circ \gamma$  and  $\gamma \circ c = c' \circ \Omega(\alpha)$ .

(LT4) (Octahedral axiom) Given two left triangles in  $\Delta$ :  $\Omega(A) \xrightarrow{f} F \xrightarrow{b} B \xrightarrow{a} A$  and  $\Omega(B) \xrightarrow{d} D \xrightarrow{c} C \xrightarrow{b'} B$ , let  $\Omega(A) \xrightarrow{e} E \xrightarrow{c'} C \xrightarrow{a \circ b'} A$  be the left triangle in  $\Delta$  corresponding to  $a \circ b'$ , then there is a commutative diagram

$$\begin{array}{ccccccc} \Omega(F) & = & \Omega(F) & & & & \\ \Omega(b) \downarrow & & \downarrow & & & & \\ \Omega(B) & \xrightarrow{d} & D & \xrightarrow{c} & C & \xrightarrow{b'} & B \\ \Omega(a) \downarrow & & \downarrow & & \parallel & & \downarrow a \\ \Omega(A) & \xrightarrow{e} & E & \xrightarrow{c'} & C & \xrightarrow{a \circ b'} & A \\ \parallel & & \downarrow & & \downarrow b' & & \parallel \\ \Omega(A) & \xrightarrow{f} & F & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

such that the second column from the left is a left triangle in  $\Delta$ .

The notion of a *right triangulated category* is defined dually.

One-sided triangulated categories as a generalization of triangulated categories arise naturally in the study of homotopy theories [18, 36, 19, 10] and derived categories [28, 26]. When the endofunctor  $\Omega$  is an autoequivalence, the left triangulated category  $(\mathcal{C}, \Omega, \Delta)$  is a triangulated category in the sense of [40]. In this paper, we will always use  $(\mathcal{T}, [1], \Delta)$  to denote a triangulated category and the quasi-inverse of  $[1]$  is denoted by  $[-1]$ .

**2.3. Partial left triangulated categories.** Let  $\mathcal{A}$  be an additive category endowed with an endofunctor  $\Omega: \mathcal{A} \rightarrow \mathcal{A}$ . We use  $\mathcal{X} \subseteq \mathcal{C} \subseteq \mathcal{A}$  to denote that  $\mathcal{X}, \mathcal{C}$  are additive subcategories of  $\mathcal{A}$  such that  $\mathcal{X}$  is a subcategory of  $\mathcal{C}$ . A morphism  $a: B \rightarrow A$  in  $\mathcal{C}$  is said to be an  $\mathcal{X}$ -*epic* if for any object  $X \in \mathcal{X}$ , the induced morphism  $a_* = \text{Hom}_{\mathcal{C}}(X, a): \text{Hom}_{\mathcal{C}}(X, B) \rightarrow \text{Hom}_{\mathcal{C}}(X, A)$  is surjective. The notion of an  $\mathcal{X}$ -*monic* is defined dually.

A sequence of the form

$$\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$$

in  $\mathcal{A}$  is called a *left  $(\mathcal{C}, \mathcal{X})$ -sequence* if the following conditions hold:

- (a)  $a$  is an  $\mathcal{X}$ -epic in  $\mathcal{C}$ ;
- (b)  $b$  is a *weak kernel* of  $a$  in  $\mathcal{C}$ , i.e., for any object  $C \in \mathcal{C}$ , the induced sequence  $\text{Hom}_{\mathcal{C}}(C, K) \xrightarrow{b_*} \text{Hom}_{\mathcal{C}}(C, B) \xrightarrow{a_*} \text{Hom}_{\mathcal{C}}(C, A)$  is exact;
- (c)  $k$  is a weak kernel of  $b$  in  $\mathcal{A}$ .

A morphism of left  $(\mathcal{X}, \mathcal{C})$ -sequences is given by a commutative diagram:

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \\ \Omega(\alpha) \downarrow & & \gamma \downarrow & & \beta \downarrow & & \downarrow \alpha \\ \Omega(A') & \xrightarrow{k'} & K' & \xrightarrow{b'} & B' & \xrightarrow{a'} & A' \end{array}$$

The composition is the obvious one.

Dually, we can define the notion of a *right  $(\mathcal{C}, \mathcal{X})$ -sequence*.

**Definition 2.2.** Let  $\mathcal{A}$  be an additive category endowed with an endofunctor  $\Omega$ . Let  $\mathcal{X} \subseteq \mathcal{C}$  be two additive subcategories of  $\mathcal{A}$  and  $\mathcal{L}(\mathcal{C}, \mathcal{X})$  a class of left  $(\mathcal{C}, \mathcal{X})$ -sequences called *left  $(\mathcal{C}, \mathcal{X})$ -triangles*. The triple  $(\mathcal{A}, \Omega, \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is said to be a *partial left triangulated category* if the following axioms hold:

(PLT1) (i) A left  $(\mathcal{C}, \mathcal{X})$ -sequence isomorphic to a left  $(\mathcal{C}, \mathcal{X})$ -triangle is itself a left  $(\mathcal{C}, \mathcal{X})$ -triangle.

(ii) The direct sum of two left  $(\mathcal{C}, \mathcal{X})$ -triangles is a left  $(\mathcal{C}, \mathcal{X})$ -triangle.

(PLT2) (i) For each object  $A \in \mathcal{C}$ , there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \rightarrow L \rightarrow X \rightarrow A$  with  $X \in \mathcal{X}$ .

(ii) For each morphism  $a: B \rightarrow A$  in  $\mathcal{C}$ ,  $\Omega(A) \xrightarrow{0} B \xrightarrow{\begin{pmatrix} -1_B \\ a \end{pmatrix}} B \oplus A \xrightarrow{(a, 1_A)} A$  is a left  $(\mathcal{C}, \mathcal{X})$ -triangle.

(iii) Given any morphism  $a: B \rightarrow A$  in  $\mathcal{C}$ , if there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \rightarrow L \rightarrow X \xrightarrow{p} A$  with  $X \in \mathcal{X}$ , then there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \rightarrow N \rightarrow B \oplus X \xrightarrow{(a,p)} A$ .

(PLT3) If the rows of the following diagram are left  $(\mathcal{C}, \mathcal{X})$ -triangles and the rightmost square is commutative, there is a morphism  $\gamma: K \rightarrow K'$  making the whole diagram commutative:

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \\ \Omega(\alpha) \downarrow & & \gamma \downarrow & & \beta \downarrow & & \downarrow \alpha \\ \Omega(A') & \xrightarrow{k'} & K' & \xrightarrow{b'} & B' & \xrightarrow{a'} & A' \end{array}$$

Moreover, if  $B \in \mathcal{X}$  and  $\alpha$  factors through  $a'$ , then any such morphism  $\gamma: K \rightarrow K'$  factors through  $b$ .

(PLT4) Let  $\Omega(B) \xrightarrow{m} M \xrightarrow{c} C \xrightarrow{b'} B$  and  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  be two left  $(\mathcal{C}, \mathcal{X})$ -left triangles such that  $\Omega(A) \xrightarrow{l} L \xrightarrow{c'} C \xrightarrow{a \circ b'} A$  is a left  $(\mathcal{C}, \mathcal{X})$ -triangle. Then there is a commutative diagram in  $\mathcal{A}$

$$\begin{array}{ccccccc} \Omega(K) & \xlongequal{\quad} & \Omega(K) & & & & \\ \Omega(b) \downarrow & & \downarrow & & & & \\ \Omega(B) & \xrightarrow{m} & M & \xrightarrow{c} & C & \xrightarrow{b'} & B \\ \Omega(a) \downarrow & & \downarrow & & \parallel & & \downarrow f \\ \Omega(A) & \xrightarrow{l} & L & \xrightarrow{c'} & C & \xrightarrow{a \circ b'} & A \\ \parallel & & \downarrow & & \downarrow b' & & \parallel \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

such that the second column from the left is a left  $(\mathcal{C}, \mathcal{X})$ -triangle.

Dually, we can define the notion of a *partial right triangulated category*.

If  $(\mathcal{A}, \Omega, \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is a partial left triangulated category, sometimes we will suppress the class  $\mathcal{L}(\mathcal{C}, \mathcal{X})$  and just say that  $(\mathcal{A}, \Omega)$  is a partial left triangulated category.

**Example 2.3.** (i) If  $(\mathcal{T}, \Omega, \Delta)$  is a left triangulated category, take  $\mathcal{X} = \mathcal{C} = \mathcal{T}$  and  $\mathcal{L}(\mathcal{T}, \mathcal{T}) = \Delta$ , then  $(\mathcal{T}, \Omega, \mathcal{L}(\mathcal{T}, \mathcal{T}))$  is a partial left triangulated category. Dually, if  $(\mathcal{T}, \Sigma, \nabla)$  is a right triangulated category, then  $(\mathcal{T}, \Sigma, \mathcal{R}(\mathcal{T}, \mathcal{T}) = \nabla)$  is a partial right triangulated category.

(ii) Let  $(\mathcal{A}, \mathcal{E})$  be an exact category (see Subsection 3.2). Let

$$\mathcal{L}(\mathcal{A}, \mathcal{A}) = \{0 \rightarrow C \rightarrow B \rightarrow A \mid 0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0 \text{ is a conflation in } \mathcal{E}\}$$

Then  $(\mathcal{A}, 0, \mathcal{L}(\mathcal{A}, \mathcal{A}))$  is a partial left triangulated category. Dually, let

$$\mathcal{R}(\mathcal{A}, \mathcal{A}) = \{C \rightarrow B \rightarrow A \rightarrow 0 \mid 0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0 \text{ is a conflation in } \mathcal{E}\}$$

Then  $(\mathcal{A}, 0, \mathcal{R}(\mathcal{A}, \mathcal{A}))$  is a partial right triangulated category.

(iii) Let  $\mathcal{A}$  be an additive category. Let  $\mathcal{L}(\mathcal{A}, \mathcal{A})$  be the class of left  $(\mathcal{A}, \mathcal{A})$ -sequences which are isomorphic to the left  $(\mathcal{A}, \mathcal{A})$ -sequences of the form  $\Omega(A) \xrightarrow{0} B \xrightarrow{(-1_B)} B \oplus A \xrightarrow{(a, 1_A)} A$ . Then  $(\mathcal{A}, 0, \mathcal{L}(\mathcal{A}, \mathcal{A}))$  is a partial left triangulated category. Dually, let  $\mathcal{R}(\mathcal{A}, \mathcal{A})$  be the class of right  $(\mathcal{A}, \mathcal{A})$ -sequences which are isomorphic to the right  $(\mathcal{A}, \mathcal{A})$ -sequences of the form  $A \xrightarrow{(1_A)} A \oplus B \xrightarrow{(a, -1_B)} B \xrightarrow{0} 0$ . Then  $(\mathcal{A}, 0, \mathcal{R}(\mathcal{A}, \mathcal{A}))$  is a partial right triangulated category.

Now suppose that  $(\mathcal{A}, \Omega, \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is a partial left triangulated category.

**Lemma 2.4.** *Let  $a: B \rightarrow A$  be an  $\mathcal{X}$ -epic in  $\mathcal{C}$ . If  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  is a left  $(\mathcal{C}, \mathcal{X})$ -triangle associating with  $a$ , then it is unique up to isomorphism.*

(ii) *Let  $\Omega(A) \xrightarrow{l} L \xrightarrow{x} X \xrightarrow{p} A$  and  $\Omega(A) \xrightarrow{l'} L' \xrightarrow{x'} X' \xrightarrow{p'} A$  be two left  $(\mathcal{C}, \mathcal{X})$ -triangles such that  $X', X \in \mathcal{X}$ . Then  $\underline{L}$  and  $\underline{L}'$  are isomorphic in  $\mathcal{C}/\mathcal{X}$ .*

*Proof.* We first show that any morphism  $u: K \rightarrow K$  in the following commutative diagram is an isomorphism:

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \\ & & \downarrow u & & \downarrow & & \downarrow \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

In fact, since  $b \circ (u - 1_K) = b \circ u - b = 0$  and  $k$  is a weak kernel of  $b$ , there is a morphism  $v: K \rightarrow \Omega(A)$  such that  $u - 1_K = k \circ v$ . Then  $(u - 1_K)^2 = (u - 1_K) \circ k \circ v = (u \circ k - k) \circ v = 0$ . Thus  $u$  is an isomorphism.

If there is another left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \xrightarrow{k'} K' \xrightarrow{b'} B \xrightarrow{a} A$ , then by (PLT3), there are morphisms  $s: K \rightarrow K'$  and  $s': K' \rightarrow K$  making the following diagram commutative:

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \\ & & \downarrow s & & \downarrow & & \downarrow \\ \Omega(A) & \xrightarrow{k'} & K' & \xrightarrow{b'} & B & \xrightarrow{a} & A \\ & & \downarrow s' & & \downarrow & & \downarrow \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

Then both  $s's$  and  $ss'$  are isomorphisms by the previous proof, thus  $s$  is an isomorphism.  $\square$

(ii) There is a commutative diagram of left  $(\mathcal{C}, \mathcal{X})$ -triangles:

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{l} & L & \xrightarrow{x} & X & \xrightarrow{p} & A \\ \parallel & & \downarrow t & & \downarrow \delta & & \parallel \\ \Omega(A) & \xrightarrow{l'} & L' & \xrightarrow{x'} & X' & \xrightarrow{p'} & A \\ \parallel & & \downarrow t' & & \downarrow \delta' & & \parallel \\ \Omega(A) & \xrightarrow{l} & L & \xrightarrow{x} & X & \xrightarrow{p} & A \end{array}$$

where the existences of  $\delta$  and  $\delta'$  are since  $p$  and  $p'$  are  $\mathcal{X}$ -epics, and the existences of  $t$  and  $t'$  are since (PLT3). Then we have the following commutative diagram

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{l} & L & \xrightarrow{x} & X & \xrightarrow{p} & A \\ 0 \downarrow & & \downarrow t' \circ t - 1_L & & \downarrow \delta' \circ \delta - 1_X & & \downarrow 0 \\ \Omega(A) & \xrightarrow{l} & L & \xrightarrow{x} & X & \xrightarrow{p} & A \end{array}$$

Thus  $t' \circ t - 1_L$  factors through  $x$  by (PLT3) since  $0 : A \rightarrow A$  factors through  $p$ . In other words  $\underline{t'} \circ \underline{t} = \underline{1}_L$ . Similarly, we can show  $\underline{t} \circ \underline{t'} = \underline{1}_{L'}$ . Thus  $\underline{L}$  is isomorphic to  $\underline{L'}$  in  $\mathcal{C}/\mathcal{X}$ .

**Lemma 2.5.** *Let  $\Omega(C) \xrightarrow{l} L \xrightarrow{x} X \xrightarrow{p} C$  and  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  be two left  $(\mathcal{C}, \mathcal{X})$ -triangles with  $X \in \mathcal{X}$ . If there is a morphism  $\alpha : C \rightarrow A$ , then there is a morphism of left  $(\mathcal{C}, \mathcal{X})$ -triangles:*

$$\begin{array}{ccccccc} \Omega(C) & \xrightarrow{l} & L & \xrightarrow{x} & X & \xrightarrow{p} & C \\ \Omega(\alpha) \downarrow & & \downarrow \gamma & & \downarrow \beta & & \downarrow \alpha \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

and  $\underline{\gamma}$  is uniquely determined by  $\underline{\alpha}$  in the stable category  $\mathcal{C}/\mathcal{X}$ .

*Proof.* The existence of  $\beta$  is since  $a$  is an  $\mathcal{X}$ -epic and  $X \in \mathcal{X}$ . The existence of  $\gamma$  is by (PLT3). Assume that there is another morphism  $\alpha' : C \rightarrow A$  such that  $\underline{\alpha'} = \underline{\alpha}$  with  $\gamma', \beta', \alpha'$  the corresponding morphisms. Then we have a commutative diagram

$$\begin{array}{ccccccc} \Omega(C) & \xrightarrow{l} & L & \xrightarrow{x} & X & \xrightarrow{p} & C \\ \Omega(\alpha) - \Omega(\alpha') \downarrow & & \downarrow \gamma - \gamma' & & \downarrow \beta - \beta' & & \downarrow \alpha - \alpha' \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$



Since  $\underline{\alpha}' = \underline{\alpha}$ , the morphism  $\alpha - \alpha'$  factors through some object in  $\mathcal{X}$ , and then it factors through  $a$  since  $a$  is an  $\mathcal{X}$ -epic. Thus  $\gamma - \gamma'$  factors through  $x$  by (PLT3). In other words,  $\underline{\gamma} = \underline{\gamma}'$  in  $\mathcal{C}/\mathcal{X}$ .  $\square$

By (PLT2)(i), for all  $A \in \mathcal{C}$ , we can choose left  $(\mathcal{C}, \mathcal{X})$ -triangles  $\Omega(A) \xrightarrow{\nu_A} U_A \xrightarrow{\iota_A} X_A \xrightarrow{p_A} A$  such that  $X_A \in \mathcal{X}$ . And then we can define a functor  $\Omega_{\mathcal{X}}: \mathcal{C}/\mathcal{X} \rightarrow \mathcal{C}/\mathcal{X}$  by sending each object  $\underline{A}$  to  $\underline{U}_A$  and each morphism  $\underline{a}: \underline{B} \rightarrow \underline{A}$  to  $\underline{\kappa}_a$ , where  $\kappa_a$  satisfies the following commutative diagram:

$$(2.6) \quad \begin{array}{ccccccc} \Omega(B) & \xrightarrow{\nu_B} & U_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{p_B} & B \\ \Omega(a) \downarrow & & \kappa_a \downarrow & & x_a \downarrow & & \downarrow a \\ \Omega(A) & \xrightarrow{\nu_A} & U_A & \xrightarrow{\iota_A} & X_A & \xrightarrow{p_A} & A \end{array}$$

By Lemma 2.5,  $\underline{\kappa}_a$  is uniquely determined by  $\underline{a}$  and thus  $\Omega_{\mathcal{X}}$  is well defined.

### 3. A HOMOTOPY THEORY OF PARTIAL LEFT TRIANGULATED CATEGORIES

In this section, we fix a partial left triangulated category  $(\mathcal{A}, \Omega, \mathcal{L}(\mathcal{C}, \mathcal{X}))$ . We call the stable category  $\mathcal{C}/\mathcal{X}$  the *homotopy category of*  $(\mathcal{A}, \Omega, \mathcal{L}(\mathcal{C}, \mathcal{X}))$ .

**3.1. From partial left triangulated categories to left triangulated categories.** If  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  is a left  $(\mathcal{C}, \mathcal{X})$ -triangle, we will use  $\dagger_a^b$  to denote it. By Lemma 2.5, if  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  is a left  $(\mathcal{C}, \mathcal{X})$ -triangle, we have a commutative diagram

$$(3.1) \quad \begin{array}{ccccccc} \Omega(A) & \xrightarrow{\nu_A} & U_A & \xrightarrow{\iota_A} & X_A & \xrightarrow{p_A} & A \\ \parallel & & \xi(\dagger_a^b) \downarrow & & \delta_a \downarrow & & \parallel \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

and the residue class of  $\underline{\xi}(\dagger_a^b)$  is uniquely determined by  $a$ . Thus  $a$  induces a sequence  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\underline{\xi}(\dagger_a^b)} \underline{K} \xrightarrow{b} \underline{B} \xrightarrow{a} \underline{A}$  in  $\mathcal{C}/\mathcal{X}$  which is said to be a *standard left triangle*. We use  $\triangle_{\mathcal{X}}$  to denote the class of left triangles in  $\mathcal{C}/\mathcal{X}$  which are isomorphic to standard left triangles. The left triangles in  $\triangle_{\mathcal{X}}$  are called *distinguished left triangles*.

Dually, if  $(\mathcal{A}, \Sigma, \mathcal{R}(\mathcal{C}, \mathcal{Y}))$  is a partial right triangulated category, we can construct an endofunctor  $\Sigma^{\mathcal{Y}}: \mathcal{C}/\mathcal{Y} \rightarrow \mathcal{C}/\mathcal{Y}$ , and the corresponding *standard right triangles*. We use  $\nabla^{\mathcal{Y}}$  to denote the class of sequences in  $\mathcal{C}/\mathcal{Y}$  which are isomorphic to standard right triangles.

We have the following theorem

**Theorem 3.2.** (i) If  $(\mathcal{A}, \Omega, \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is a partial left triangulated category, then  $(\mathcal{C}/\mathcal{X}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$  is a left triangulated category.

(ii) If  $(\mathcal{A}, \Sigma, \mathcal{R}(\mathcal{C}, \mathcal{Y}))$  is a partial right triangulated category, then  $(\mathcal{C}/\mathcal{Y}, \Sigma^{\mathcal{Y}}, \nabla^{\mathcal{Y}})$  is a right triangulated category.

This theorem gives a *homotopy theory* of a partial one-sided triangulated category (the homotopy category of a partial one-sided triangulated category together with the corresponding one-sided triangulated structures).

Before the proof we need two Lemmas.

**Lemma 3.3.** Assume that we have a commutative diagram of left  $(\mathcal{C}, \mathcal{X})$ -triangles in  $\mathcal{A}$

$$\begin{array}{ccccccc} \Omega(C) & \xrightarrow{m} & M & \xrightarrow{d} & D & \xrightarrow{c} & C \\ \Omega(\alpha) \downarrow & & \gamma \downarrow & & \beta \downarrow & & \downarrow \alpha \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

Then  $\underline{\gamma} \circ \underline{\xi}(\dagger_c^d) = \underline{\xi}(\dagger_a^b) \circ \underline{\kappa}_{\alpha}$  in  $\mathcal{C}/\mathcal{X}$ .

*Proof.* By assumption and (3.1), we have the following diagram

$$\begin{array}{ccccccc} \Omega(C) & \xrightarrow{\nu_C} & U_C & \xrightarrow{\iota_C} & X_C & \xrightarrow{p_C} & C \\ \parallel & & \xi(\dagger_c^d) \downarrow & & \delta_c \downarrow & & \parallel \\ \Omega(C) & \xrightarrow{m} & M & \xrightarrow{d} & D & \xrightarrow{c} & C \\ \Omega(\alpha) \downarrow & & \gamma \downarrow & & \beta \downarrow & & \downarrow \alpha \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

Similarly, by (2.6) and (3.1), we have the following commutative diagram

$$\begin{array}{ccccccc} \Omega(C) & \xrightarrow{\nu_C} & U_C & \xrightarrow{\iota_C} & X_C & \xrightarrow{p_C} & C \\ \Omega(\alpha) \downarrow & & \kappa_{\alpha} \downarrow & & x_{\alpha} \downarrow & & \downarrow \alpha \\ \Omega(A) & \xrightarrow{\nu_A} & U_A & \xrightarrow{\iota_A} & X_A & \xrightarrow{p_A} & A \\ \parallel & & \xi(\dagger_a^b) \downarrow & & \delta_a \downarrow & & \parallel \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

Thus the assertion follows from Lemma 2.5 directly.  $\square$

Given any morphism  $a: B \rightarrow A$  in  $\mathcal{C}$ , by (PLT2)(iii), there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \xrightarrow{n} N \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} B \oplus X_A \xrightarrow{(a, p_A)} A$ . Let  $\Omega(A) \xrightarrow{\nu_A} K_A \xrightarrow{\iota_A} X_A \xrightarrow{p_A} A$  be the chosen  $(\mathcal{C}, \mathcal{X})$ -left triangle for  $A$ , then  $\Omega(B) \oplus \Omega(A) \xrightarrow{(0, \nu_A)} K_A \xrightarrow{\begin{pmatrix} 0 \\ \iota_A \end{pmatrix}} B \oplus X_A \xrightarrow{\begin{pmatrix} 1_B & 0 \\ 0 & p_A \end{pmatrix}} B \oplus A$

is a left  $(\mathcal{C}, \mathcal{X})$ -triangle by (PLT1)(ii) and (PLT2)(ii). So we can get the following commutative diagram by (PLT4):

$$(3.4) \quad \begin{array}{ccccccc} \Omega(B) & \xlongequal{\quad} & \Omega(B) & & & & \\ \downarrow \begin{pmatrix} -1_{\Omega(B)} \\ \Omega(a) \end{pmatrix} & & \downarrow & & \downarrow \begin{pmatrix} 0 \\ \iota_A \end{pmatrix} & & \downarrow \begin{pmatrix} 1_B & 0 \\ 0 & p_A \end{pmatrix} \\ \Omega(B) \oplus \Omega(A) & \xrightarrow{(0, \nu_A)} & K_A & \xrightarrow{\quad} & B \oplus X_A & \xrightarrow{\quad} & B \oplus A \\ \downarrow (\Omega(a), 1_{\Omega(A)}) & & \downarrow \zeta & & \parallel & & \downarrow (a, 1_A) \\ \Omega(A) & \xrightarrow{n} & N & \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} & B \oplus X_A & \xrightarrow{(a, p_A)} & A \\ \parallel & & \downarrow \eta & & \downarrow \begin{pmatrix} 1_B & 0 \\ 0 & p_A \end{pmatrix} & & \parallel \\ \Omega(A) & \xrightarrow{0} & B & \xrightarrow{\begin{pmatrix} -1_B \\ a \end{pmatrix}} & B \oplus A & \xrightarrow{(a, 1_A)} & A \end{array}$$

such that the second column from the left is a left  $(\mathcal{C}, \mathcal{X})$ -triangle. By Lemma 3.3, we have  $\underline{\xi}(\dagger_\eta^\zeta) = \underline{\xi}(\dagger_{\begin{pmatrix} \iota_A \\ 1_B & 0 \\ 0 & p_A \end{pmatrix}}) \circ \underline{\kappa} \begin{pmatrix} -1_B \\ f \end{pmatrix} = (0, 1_{K_A}) \circ \begin{pmatrix} -1_{K_B} \\ \underline{\kappa}_a \end{pmatrix} = \underline{\kappa}_a$ , where  $\underline{\xi}(\dagger_\eta^\zeta)$  and  $\kappa_a$  are defined as in (3.1) and (2.6).

**Lemma 3.5.** (i)  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\underline{\zeta}} \underline{N} \xrightarrow{-\eta} \underline{B} \xrightarrow{a} \underline{A}$  is a standard left triangle.

(ii)  $\Omega_{\mathcal{X}}(\underline{B}) \xrightarrow{\Omega_{\mathcal{X}}(a)} \Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\underline{\zeta}} \underline{N} \xrightarrow{\eta} \underline{B}$  is a standard left triangle.

(iii) If  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  is a left  $(\mathcal{C}, \mathcal{X})$ -left triangle, then the standard left triangle  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\underline{\xi}(\dagger_a^b)} \underline{K} \xrightarrow{b} \underline{B} \xrightarrow{a} \underline{A}$  is isomorphic to the standard left triangle  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\underline{\zeta}} \underline{N} \xrightarrow{-\eta} \underline{B} \xrightarrow{a} \underline{A}$  induced from  $(a, p_A)$ .

*Proof.* (i) By (3.4), there is a commutative diagram of left  $(\mathcal{X}, \mathcal{C})$ -triangles:

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{\nu_A} & U_A & \xrightarrow{\iota_A} & X_A & \xrightarrow{p_A} & A \\ \parallel & & \downarrow \zeta & & \downarrow \begin{pmatrix} 0 \\ 1_{X_A} \end{pmatrix} & & \parallel \\ \Omega(A) & \xrightarrow{n} & N & \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} & B \oplus X_A & \xrightarrow{(a, p_A)} & A \end{array}$$

Thus  $\underline{\zeta} = \underline{\xi}(\dagger_{(a, p_A)}^{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}})$  by Lemma 2.5 and  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\underline{\zeta}} \underline{N} \xrightarrow{-\eta} \underline{B} \xrightarrow{a} \underline{A}$  is a standard left triangle.

(ii) This follows from (3.4).

(iii) Let  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  and  $\Omega(A) \xrightarrow{n} N \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} B \oplus X_A \xrightarrow{(a, p_A)} A$  be two left  $(\mathcal{C}, \mathcal{X})$ -triangles. Consider the following diagram of left  $(\mathcal{C}, \mathcal{X})$ -triangles:

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{n} & N & \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} & B \oplus X_A & \xrightarrow{(a, p_A)} & A \\ \parallel & & \downarrow t & & \downarrow (1_B, \delta_a) & & \parallel \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

where  $\delta_a: X_A \rightarrow B$  is constructed in (3.1) which satisfies  $a \circ \delta_a = p_A$ . Thus we have  $a \circ (1_B, \delta_a) = (a, p_A)$  and then there exists a morphism  $t: N \rightarrow K$  making the above whole diagram commutative by (PLT3). Thus  $\underline{\xi}(\dagger_a^b) = \underline{t} \circ \underline{\xi}(\dagger_{(a, p_A)}^{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}}) = \underline{t} \circ \underline{\zeta}$  in  $\mathcal{C}/\mathcal{X}$  by Lemma 3.3. So we have a commutative diagram of standard left triangles in  $\mathcal{C}/\mathcal{X}$ :

$$\begin{array}{ccccccc} \Omega_{\mathcal{X}}(\underline{A}) & \xrightarrow{\underline{\zeta}} & \underline{N} & \xrightarrow{-\eta} & \underline{B} & \xrightarrow{a} & \underline{A} \\ \parallel & & \downarrow \underline{t} & & \parallel & & \parallel \\ \Omega_{\mathcal{X}}(\underline{A}) & \xrightarrow{\underline{\xi}(\dagger_a^b)} & \underline{K} & \xrightarrow{\underline{b}} & \underline{B} & \xrightarrow{a} & \underline{A} \end{array}$$

We will show that  $\underline{t}$  is an isomorphism. In fact, by (PLT3), there is a morphism  $\tau: K \rightarrow N$  such that the following diagram of left  $(\mathcal{C}, \mathcal{X})$ -triangles commutative

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \\ \parallel & & \downarrow \tau & & \downarrow \begin{pmatrix} -\eta \\ \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \parallel \\ \Omega(A) & \xrightarrow{n} & N & \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} & B \oplus X_A & \xrightarrow{(a, p_A)} & A \end{array}$$

It can be shown that  $t \circ \tau$  is an isomorphism by the proof of Lemma 2.4. Since  $(a, p_A) \circ \begin{pmatrix} -\delta_a \\ 1_{X_A} \end{pmatrix} = p_A - a \circ \delta_a = 0$  by (3.1), there is a morphism  $x: X_A \rightarrow N$  such that  $\begin{pmatrix} -\eta \\ \theta \end{pmatrix} \circ x = \begin{pmatrix} -\delta_a \\ 1_{X_A} \end{pmatrix}$ . Then we have the following commutative diagram of left  $(\mathcal{C}, \mathcal{X})$ -triangles

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{n} & N & \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} & B \oplus X_A & \xrightarrow{(a, p_A)} & A \\ \parallel & & \downarrow \tau \circ t + x \circ \theta & & \parallel & & \parallel \\ \Omega(A) & \xrightarrow{n} & N & \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} & B \oplus X_A & \xrightarrow{(a, p_A)} & A \end{array}$$

So  $\tau \circ t + x \circ \theta$  is an isomorphism by the proof of Lemma 2.4, and thus  $\underline{\tau} \circ \underline{t}$  is an isomorphism. Therefore  $\underline{t}$  is an isomorphism and we are done.  $\square$

### 3.2. The proof of Theorem 3.2.

*Proof.* (i) We verify (LT1)-(LT4) of Definition 2.1 one by one.

(LT1) For each object  $A \in \mathcal{C}$ , there is a standard left triangle  $0 \rightarrow \underline{A} \xrightarrow{1_A} \underline{A} \rightarrow 0$  induced by the left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(0) = 0 \rightarrow A \xrightarrow{1_A} A \rightarrow 0$ . Given any morphism  $a: B \rightarrow A$ , by (PLT2)(iii), there is a left  $(\mathcal{C}, \mathcal{X})$ -left triangle  $\Omega(A) \xrightarrow{n} N \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} B \oplus X_A \xrightarrow{(a, p_A)} A$  which induces a standard left triangle  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\zeta} \underline{N} \xrightarrow{b} \underline{B} \xrightarrow{a} \underline{A}$  by Lemma 3.5 (i).

(LT2) Let  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\zeta} \underline{N} \xrightarrow{-\eta} \underline{B} \xrightarrow{a} \underline{A}$  be a standard left triangle induced by the left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \xrightarrow{n} N \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} B \oplus X_A \xrightarrow{(a, p_A)} A$ . Then  $\Omega_{\mathcal{X}}(\underline{B}) \xrightarrow{-\Omega_{\mathcal{X}}(\underline{a})} \Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\zeta} \underline{N} \xrightarrow{-\eta} \underline{B}$  is a distinguished left triangle since it is isomorphic to the standard left triangle  $\Omega_{\mathcal{X}}(\underline{B}) \xrightarrow{\Omega_{\mathcal{X}}(\underline{a})} \Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\zeta} \underline{N} \xrightarrow{\eta} \underline{B}$  (see Lemma 3.5 (ii)) via the triple  $(1_{\Omega_{\mathcal{X}}(\underline{A})}, 1_{\underline{N}}, -1_{\underline{B}})$ .

(LT3) Assume that we have a diagram of standard left triangles

$$\begin{array}{ccccccc} \Omega_{\mathcal{X}}(\underline{C}) & \xrightarrow{\xi(\dagger_c^d)} & \underline{M} & \xrightarrow{d} & \underline{D} & \xrightarrow{c} & \underline{C} \\ \Omega_{\mathcal{X}}(\underline{\alpha}) \downarrow & & & & \downarrow \beta & & \downarrow \alpha \\ \Omega_{\mathcal{X}}(\underline{A}) & \xrightarrow{\xi(\dagger_a^b)} & \underline{K} & \xrightarrow{b} & \underline{B} & \xrightarrow{a} & \underline{A} \end{array}$$

with the rightmost square commutative. Let  $\Omega(C) \xrightarrow{m} M \xrightarrow{d} D \xrightarrow{c} C$  and  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  be the corresponding left  $(\mathcal{C}, \mathcal{X})$ -triangles respectively. Since  $\underline{\alpha} \circ \underline{c} = \underline{a} \circ \underline{\beta}$ , there is a morphism  $l: D \rightarrow X_A$  such that  $\alpha \circ c - a \circ \beta = p_A \circ l$ . By (3.1), there is a morphism  $\delta_a: X_A \rightarrow B$  such that  $a \circ \delta_a = p_A$  and thus  $a \circ (\beta + \delta_a \circ l) = a \circ \beta + a \circ \delta_a \circ l = a \circ \beta + p_A \circ l = \alpha \circ c$ . So by (PLT3), there is a morphism  $s: M \rightarrow K$  making the following diagram of left  $(\mathcal{C}, \mathcal{X})$ -triangles commutative

$$\begin{array}{ccccccc} \Omega(C) & \xrightarrow{m} & M & \xrightarrow{d} & D & \xrightarrow{c} & C \\ \Omega_{\mathcal{X}} \downarrow & & \vdots s & & \downarrow \beta + \delta_a \circ l & & \downarrow \alpha \\ \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

Then  $\underline{s} \circ \xi(\dagger_c^d) = \xi(\dagger_a^b) \circ \Omega_{\mathcal{X}}(\underline{\alpha})$  by Lemma 3.3. Therefore  $\underline{s}: \underline{M} \rightarrow \underline{K}$  is the desired filler.

(LT4). Assume that we have two standard left triangles  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\xi(\dagger_c^{b'})} \underline{L} \xrightarrow{c} \underline{C} \xrightarrow{b'} \underline{B}$  and  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\zeta} \underline{N} \xrightarrow{-\eta} \underline{B} \xrightarrow{a} \underline{A}$  induced by the left  $(\mathcal{C}, \mathcal{X})$ -triangles  $\Omega(B) \xrightarrow{l} L \xrightarrow{c} C \xrightarrow{b'} B$  and  $\Omega(A) \xrightarrow{n} N \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} B \oplus X_A \xrightarrow{(a, p_A)} A$  respectively. By (PLT1)(ii) and (PLT2)(ii),

$\Omega(B) \oplus \Omega(X_A) \xrightarrow{(l,0)} L \xrightarrow{\begin{pmatrix} c \\ 0 \end{pmatrix}} C \oplus X_A \xrightarrow{\begin{pmatrix} b' & 0 \\ 0 & 1_{X_A} \end{pmatrix}} B \oplus X_A$  is a left  $(\mathcal{C}, \mathcal{X})$ -triangle. By (PLT2)(iii), there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \xrightarrow{m} M \xrightarrow{\begin{pmatrix} -\lambda \\ \chi \end{pmatrix}} C \oplus X_A \xrightarrow{(a \circ b', p_A)} A$ . So there is a commutative diagram in  $\mathcal{A}$  by (PLT4):

$$(3.6) \quad \begin{array}{ccccccc} \Omega(N) & \xlongequal{\quad} & \Omega(N) & & & & \\ \downarrow \begin{pmatrix} -\Omega(\eta) \\ \Omega(\theta) \end{pmatrix} & & \downarrow & & \downarrow \begin{pmatrix} c \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} b' & 0 \\ 0 & 1_{X_A} \end{pmatrix} \\ \Omega(B) \oplus \Omega(X_A) & \xrightarrow{(l,0)} & L & \xrightarrow{\begin{pmatrix} c \\ 0 \end{pmatrix}} & C \oplus X_A & \xrightarrow{\begin{pmatrix} b' & 0 \\ 0 & 1_{X_A} \end{pmatrix}} & B \oplus X_A \\ \downarrow (\Omega(a), \Omega(p_A)) & & \downarrow \alpha & & \downarrow \parallel & & \downarrow (a, p_A) \\ \Omega(A) & \xrightarrow{m} & M & \xrightarrow{\begin{pmatrix} -\lambda \\ \chi \end{pmatrix}} & C \oplus X_A & \xrightarrow{(a \circ b', p_A)} & A \\ \parallel & & \downarrow \beta & & \downarrow \begin{pmatrix} b' & 0 \\ 0 & 1_{X_A} \end{pmatrix} & & \parallel \\ \Omega(A) & \xrightarrow{n} & N & \xrightarrow{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}} & B \oplus X_A & \xrightarrow{(a, p_A)} & A \end{array}$$

such that the second column from the left is a  $(\mathcal{C}, \mathcal{X})$ -left triangle. By Lemma 3.5 (i), there are standard left triangles  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\zeta} \underline{M} \xrightarrow{-\lambda} \underline{C} \xrightarrow{a \circ b'} \underline{A}$  and  $\Omega_{\mathcal{X}}(\underline{A}) \xrightarrow{\zeta} \underline{N} \xrightarrow{-\eta} \underline{B} \xrightarrow{a} \underline{A}$ . So the above commutative diagram induces a diagram of left triangles in  $\mathcal{C}/\mathcal{X}$ :

$$\begin{array}{ccccccc} \Omega_{\mathcal{X}}(\underline{N}) & \xlongequal{\quad} & \Omega_{\mathcal{X}}(\underline{N}) & & & & \\ \downarrow \Omega_{\mathcal{X}}(-\eta) & & \downarrow \xi(\dagger_{\beta}^{\alpha}) & & \downarrow \xi(\dagger_{b'}^c) & & \downarrow \xi(\dagger_{b'}^c) \\ \Omega_{\mathcal{X}}(\underline{B}) & \xrightarrow{\underline{\zeta}} & \underline{L} & \xrightarrow{\underline{\varepsilon}} & \underline{C} & \xrightarrow{\underline{b}'} & \underline{B} \\ \downarrow \Omega_{\mathcal{X}}(\underline{a}) & & \downarrow \underline{\alpha} & & \downarrow \parallel & & \downarrow \underline{a} \\ \Omega_{\mathcal{X}}(\underline{A}) & \xrightarrow{\underline{\zeta}} & \underline{M} & \xrightarrow{-\lambda} & \underline{C} & \xrightarrow{a \circ b'} & \underline{A} \\ \parallel & & \downarrow \underline{\beta} & & \downarrow \underline{b}' & & \parallel \\ \Omega_{\mathcal{X}}(\underline{A}) & \xrightarrow{\underline{\zeta}} & \underline{N} & \xrightarrow{-\eta} & \underline{B} & \xrightarrow{\underline{a}} & \underline{A} \end{array}$$

Since the middle and the right hand squares are commutative, to finish the proof of (LT4), we have to show that  $\underline{\xi}(\dagger_{\beta}^{\alpha}) = \underline{\xi}(\dagger_{b'}^c) \circ \Omega_{\mathcal{X}}(-\eta)$ ,  $\underline{\beta} \circ \underline{\varepsilon} = \underline{\zeta}$  and  $\underline{\alpha} \circ \underline{\xi}(\dagger_{b'}^c) = \underline{\varepsilon} \circ \Omega_{\mathcal{X}}(\underline{a})$ .

By (3.6), we have a commutative diagram of left  $(\mathcal{C}, \mathcal{X})$ -triangles

$$\begin{array}{ccccccc} \Omega(N) & \xrightarrow{-l \circ \Omega(\eta)} & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N \\ \downarrow \Omega(\eta) & & \parallel & & \downarrow -\lambda & & \downarrow -\eta \\ \Omega(B) & \xrightarrow{l} & L & \xrightarrow{c} & C & \xrightarrow{b'} & B \end{array}$$

Thus  $\underline{\xi}(\dagger_{\beta}^{\alpha}) = \underline{\xi}(\dagger_{b'}^c) \circ \Omega_{\mathcal{X}}(-\eta)$  by Lemma 3.3. Similarly we can prove that  $\underline{\beta} \circ \underline{\varepsilon} = \underline{\zeta}$  and  $\underline{\alpha} \circ \underline{\xi}(\dagger_{b'}^c) = \underline{\varepsilon} \circ \Omega_{\mathcal{X}}(\underline{a})$  by noting that  $\underline{\varepsilon} = \underline{\xi}(\dagger_{(a \circ b', p_A)}^{\begin{pmatrix} -\lambda \\ \chi \end{pmatrix}})$  and  $\underline{\zeta} = \underline{\xi}(\dagger_{(a, p_A)}^{\begin{pmatrix} -\eta \\ \theta \end{pmatrix}})$ .

The statement (ii) can be proved dually.  $\square$

#### 4. EXAMPLES OF PARTIAL ONE-SIDED TRIANGULATED CATEGORIES

##### 4.1. Partial one-sided triangulated categories from additive categories.

**Proposition 4.1.** *Let  $\mathcal{C}$  be an additive category and  $\mathcal{X}$  an additive subcategory of  $\mathcal{C}$ .*

(i) *Assume the following conditions hold:*

(a) *For each object  $A \in \mathcal{C}$ , there is an  $\mathcal{X}$ -epic  $p: X \rightarrow A$  such that  $X \in \mathcal{X}$  and  $p$  has a kernel.*

(b) *Any morphism of the form  $(a, p'): B \oplus X' \rightarrow A$  has a kernel, where  $X' \in \mathcal{X}$  and  $p'$  is an  $\mathcal{X}$ -epic admitting a kernel.*

*Let  $\mathcal{L}(\mathcal{C}, \mathcal{X})$  be the class of all left  $(\mathcal{C}, \mathcal{X})$ -sequences. Then  $(\mathcal{C}, 0, \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is a partial left triangulated category.*

(ii) *Assume the following conditions hold:*

(c) *For each object  $A \in \mathcal{C}$ , there is an  $\mathcal{X}$ -monic  $i: A \rightarrow X$  such that  $X \in \mathcal{X}$  and  $i$  has a cokernel.*

(d) *Any morphism of the form  $(i', ) : A \rightarrow X' \oplus B$  has a cokernel, where  $X' \in \mathcal{X}$  and  $i'$  is an  $\mathcal{X}$ -monic admitting a cokernel.*

*Let  $\mathcal{R}(\mathcal{C}, \mathcal{X})$  be the class of all right  $(\mathcal{C}, \mathcal{X})$ -sequences. Then  $(\mathcal{C}, 0, \mathcal{R}(\mathcal{C}, \mathcal{X}))$  is a partial right triangulated category.*

*Proof.* We only prove (i), the statement (ii) can be proved dually.

(i) In this case, a left  $(\mathcal{C}, \mathcal{X})$ -sequence is just a *kernel sequence*  $0 \rightarrow K \xrightarrow{b} B \xrightarrow{a} A$  in  $\mathcal{C}$  with  $a$  an  $\mathcal{X}$ -epic. We will show that the class of all left  $(\mathcal{C}, \mathcal{X})$ -sequences satisfies (PLT1)-(PLT4).

(PLT1) We only need to show that the direct sum of two left  $(\mathcal{C}, \mathcal{X})$ -sequences is again a left  $(\mathcal{C}, \mathcal{X})$ -sequence. This follows from the fact that both the class of  $\mathcal{X}$ -epics and the class of kernel sequences are closed under direct sums.

(PLT2) (i) By the condition (a), for each object  $A \in \mathcal{C}$ , there is an  $\mathcal{X}$ -epic  $p_A: X_A \rightarrow A$  with  $X_A \in \mathcal{X}$  such that  $p_A$  has a kernel. In other words, there is a left  $(\mathcal{C}, \mathcal{X})$ -sequence  $0 \rightarrow K_A \xrightarrow{\iota_A} X_A \xrightarrow{p_A} A$  with  $X_A \in \mathcal{X}$ .

(ii) For each morphism  $a: B \rightarrow A$ , the sequence  $0 \rightarrow B \xrightarrow{\begin{pmatrix} -1_B \\ a \end{pmatrix}} B \oplus A \xrightarrow{(a, 1_A)} A$  is a left  $(\mathcal{C}, \mathcal{X})$ -sequence.

(iii) For any morphism  $a: B \rightarrow A$ , if  $p: X \rightarrow A$  is an  $\mathcal{X}$ -epic such that  $X \in \mathcal{X}$  and  $p$  has a kernel, then the morphism  $(a, p): B \oplus X \rightarrow A$  has a kernel by the assumption (b), i.e., there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $0 \rightarrow N \rightarrow B \oplus X \xrightarrow{(a, p)} A$ .

(PLT3) Assume that we have a diagram of left  $(\mathcal{C}, \mathcal{X})$ -sequences with the right-most square being commutative

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{b} & B & \xrightarrow{a} & A \\ & & \gamma \downarrow & & \beta \downarrow & & \downarrow \alpha \\ 0 & \longrightarrow & K' & \xrightarrow{b'} & B' & \xrightarrow{a'} & A' \end{array}$$

Since  $b' \circ (\beta \circ b) = \alpha \circ (a \circ b) = 0$ , there is a morphism  $\gamma: K \rightarrow K'$  making the whole diagram commutative. Moreover, assume that  $B \in \mathcal{X}$  and there is a morphism  $\gamma: K \rightarrow K'$  such that the above whole diagram is commutative. If  $\alpha$  factors through  $a'$ , i.e. there is a morphism  $s: A \rightarrow B'$  such that  $\alpha = b' \circ s$ , then  $a' \circ (\beta - s \circ a) = a' \circ \beta - a' \circ s \circ a = a' \circ \beta - \alpha \circ a = 0$ . Thus there is a morphism  $t: B \rightarrow K'$  such that  $\beta - s \circ a = b' \circ t$  since  $b'$  is a kernel of  $a'$ . Therefore, we have  $b' \circ \gamma = \beta \circ b = \beta \circ b - s \circ 0 = \beta \circ b - s \circ (a \circ b) = (\beta - s \circ a) \circ b = b' \circ t \circ b$ . So we have  $\gamma = t \circ b$  since  $b'$  is a monomorphism.

(PLT4) Let  $0 \rightarrow K \xrightarrow{b} B \xrightarrow{a} A$  and  $0 \rightarrow L \xrightarrow{c} C \xrightarrow{b'} B$  be two left  $(\mathcal{C}, \mathcal{X})$ -triangles such that there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $0 \rightarrow M \xrightarrow{c'} C \xrightarrow{a \circ b'} A$ . Then we have the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & L & \xrightarrow{c} & C & \xrightarrow{b'} & B \\ & & \alpha \downarrow & & \parallel & & \downarrow a \\ 0 & \longrightarrow & M & \xrightarrow{c'} & C & \xrightarrow{a \circ b'} & A \\ & & \beta \downarrow & & \downarrow b' & & \parallel \\ 0 & \longrightarrow & K & \xrightarrow{b} & B & \xrightarrow{a} & A \end{array}$$

such that  $\beta$  is an  $\mathcal{X}$ -epic and the second column from the left is a kernel sequence by [7, Lemma 2.11], i.e.,  $0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} K$  is a left  $(\mathcal{C}, \mathcal{X})$ -sequence.  $\square$

*Remark 4.2.* (i) Let  $\mathcal{C}$  be an additive category and  $\mathcal{X}$  an additive subcategory of  $\mathcal{C}$ . Recall that  $\mathcal{X}$  is said to be *contravariantly finite* in  $\mathcal{C}$  if each object  $C$  of  $\mathcal{C}$  has a *right  $\mathcal{X}$ -approximation*, i.e., there is an  $\mathcal{X}$ -epic  $X_C \rightarrow C$  with  $X_C \in \mathcal{X}$ ; see [2]. So if  $\mathcal{X}$  is contravariantly finite in  $\mathcal{C}$  and each  $\mathcal{X}$ -epic has a kernel, then  $(\mathcal{C}, 0, \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is a partial left triangulated category with  $\mathcal{L}(\mathcal{C}, \mathcal{X})$  the class of all left  $(\mathcal{C}, \mathcal{X})$ -sequences.



Dually, if  $\mathcal{X}$  is *covariantly finite* in  $\mathcal{C}$  and each  $\mathcal{X}$ -monic has a cokernel, then  $(\mathcal{C}, 0, \mathcal{C}, \mathcal{X})$  is a partial right triangulated category with  $\mathcal{R}(\mathcal{C}, \mathcal{X})$  the class of all right  $(\mathcal{C}, \mathcal{X})$ -sequences.

(ii) By Proposition 4.1, we know that we can obtain [32, Theorem 3.7] and [7, Theorem 2.12] directly by our Theorem 3.2.

**4.2. Partial one-sided triangulated categories from exact categories.** Let  $\mathcal{A}$  be an additive category. A *kernel-cokernel sequence* in  $\mathcal{A}$  is sequence  $0 \rightarrow C \xrightarrow{i} B \xrightarrow{d} A \rightarrow 0$  such that  $i = \ker d$  and  $d = \operatorname{coker} i$ . Let  $\mathcal{E}$  be a class of kernel-cokernel sequences of  $\mathcal{A}$ . Following Keller [25, Appendix A], we call a kernel-cokernel sequence a *conflation* if it is in  $\mathcal{E}$ . A morphism  $i$  is called an *inflation* if there is a conflation  $0 \rightarrow C \xrightarrow{i} B \xrightarrow{d} A \rightarrow 0$  and the morphism  $d$  is called a *deflation*.

Recall that an *exact structure* on an additive category  $\mathcal{A}$  is a class  $\mathcal{E}$  of kernel-cokernel sequences which is closed under isomorphisms and satisfies the following axioms due to Quillen [37] and Keller [25, Appendix A]:

- (Ex0) the identity morphism of the zero object is an inflation.
- (Ex1) The class of deflations is closed under composition.
- (Ex1)<sup>op</sup> The class of inflations is closed under composition.
- (Ex2) For any deflation  $d: B \rightarrow A$  and any morphism  $f: A' \rightarrow A$ , there exists a pullback diagram such that  $d'$  is a deflation:

$$\begin{array}{ccc} B' & \xrightarrow{d'} & A' \\ f' \downarrow & & \downarrow f \\ B & \xrightarrow{d} & A \end{array}$$

- (Ex2)<sup>op</sup> For any inflation  $i: C \rightarrow B$  and any morphism  $g: C \rightarrow C'$ , there is a pushout diagram such that  $i'$  is an inflation:

$$\begin{array}{ccc} C & \xrightarrow{i} & B \\ g \downarrow & & \downarrow g' \\ C' & \xrightarrow{i'} & B' \end{array}$$

An *exact category* is a pair  $(\mathcal{A}, \mathcal{E})$  consisting of an additive category  $\mathcal{A}$  and an exact structure  $\mathcal{E}$ . We refer the reader to [9, Definition 2.1] for a readable introduction to exact categories. Sometimes we suppress the class of  $\mathcal{E}$  and just say that  $\mathcal{A}$  is an exact category.

In an exact category  $(\mathcal{A}, \mathcal{E})$ , we can define the Yoneda Ext bifunctor  $\operatorname{Ext}_{\mathcal{A}}^1(C, A)$ . It is the abelian group of equivalence classes of conflations  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  in  $\mathcal{E}$ ; see [31, Chapter XII.4] for details.

If  $(\mathcal{A}, \mathcal{E})$  is an exact category and  $\mathcal{X}$  is an additive subcategory of  $\mathcal{A}$ , we call a morphism  $d: B \rightarrow A$  an  $\mathcal{X}$ -epic deflation if it is both an  $\mathcal{X}$ -epic and a deflation. Dually, we can define the notion of an  $\mathcal{X}$ -monic inflation.

**Proposition 4.3.** *Let  $(\mathcal{A}, \mathcal{E})$  be an exact category and  $\mathcal{X}$  an additive subcategory of  $\mathcal{A}$ .*

(i) *If every object  $A \in \mathcal{A}$ , there is an  $\mathcal{X}$ -epic deflation  $X \xrightarrow{p} A$  with  $X \in \mathcal{X}$ , then  $(\mathcal{A}, 0, \mathcal{L}(\mathcal{A}, \mathcal{X}))$  is a partial left triangulated category with  $\mathcal{L}(\mathcal{A}, \mathcal{X}) = \{0 \rightarrow K \rightarrow B \xrightarrow{d} A \mid 0 \rightarrow K \rightarrow B \xrightarrow{d} A \rightarrow 0 \in \mathcal{E} \text{ with } d \text{ an } \mathcal{X}\text{-epic}\}$ .*

(ii) *If every object  $A \in \mathcal{A}$ , there is an  $\mathcal{X}$ -monic inflation  $A \xrightarrow{i} X$  with  $X \in \mathcal{X}$ , then  $(\mathcal{A}, \mathcal{X}; \mathcal{A}, 0, \mathcal{R}(\mathcal{A}, \mathcal{X}))$  is a partial right triangulated category with  $\mathcal{R}(\mathcal{A}, \mathcal{X}) = \{K \xrightarrow{i} B \rightarrow A \rightarrow 0 \mid 0 \rightarrow K \xrightarrow{i} B \rightarrow A \rightarrow 0 \in \mathcal{E} \text{ with } i \text{ an } \mathcal{X}\text{-monic}\}$ .*

*Proof.* We only prove (i), the statement (ii) can be proved dually. We will show that the class  $\mathcal{L}(\mathcal{A}, \mathcal{X})$  satisfies (PLT1)-(PLT4).

(PLT1) The statements (i) and (ii) follow from the fact that both the class of  $\mathcal{X}$ -epics and the class of conflations are closed under isomorphisms and direct sums.

(PLT2) (i) By assumption, for each object  $A \in \mathcal{A}$ , there is a left  $(\mathcal{A}, \mathcal{X})$ -triangle  $0 \rightarrow L \xrightarrow{\iota} X \xrightarrow{p} A$  in  $\mathcal{L}(\mathcal{A}, \mathcal{X})$  with  $X \in \mathcal{X}$  and  $p$  an  $\mathcal{X}$ -epic deflation.

(ii) For each morphism  $a: B \rightarrow A$ , by [9, Proposition 2.12],  $0 \rightarrow B \xrightarrow{\begin{pmatrix} -1_B \\ a \end{pmatrix}} B \oplus A \xrightarrow{\begin{pmatrix} a & 1_A \end{pmatrix}} A$  is in  $\mathcal{L}(\mathcal{A}, \mathcal{X})$ .

(iii) Let  $0 \rightarrow L \rightarrow X' \xrightarrow{p'} A \rightarrow 0$  be a conflation with  $p'$  an  $\mathcal{X}$ -epic and  $X' \in \mathcal{X}$ . Then for any morphism  $a: B \rightarrow A$ , by (EX2), we have a pullback diagram

$$\begin{array}{ccc} M & \xrightarrow{\eta} & B \\ \theta \downarrow & & \downarrow a \\ X' & \xrightarrow{p'} & A \end{array}$$

Thus  $0 \rightarrow M \rightarrow B \oplus X' \xrightarrow{(a, p')} A$  is in  $\mathcal{L}(\mathcal{A}, \mathcal{X})$  by the dual of [9, Proposition 2.12].

(PLT3) See the proof of (PLT3) of Proposition 4.1.

(PLT4) Let  $0 \rightarrow K \xrightarrow{b} B \xrightarrow{a} A$  and  $0 \rightarrow L \xrightarrow{c} C \xrightarrow{b'} B$  be two left  $(\mathcal{X}, \mathcal{C})$ -triangles such that there is a left  $(\mathcal{A}, \mathcal{X})$ -triangle  $0 \rightarrow M \xrightarrow{c'} C \xrightarrow{aob'} A$ . Then we have the

following commutative diagram in  $\mathcal{A}$

$$\begin{array}{ccccccc}
& & 0 & & & & \\
& & \downarrow & & & & \\
0 & \longrightarrow & L & \xrightarrow{c} & C & \xrightarrow{b'} & B \longrightarrow 0 \\
& & \alpha \downarrow & & \parallel & & \downarrow a \\
0 & \longrightarrow & M & \xrightarrow{c'} & C & \xrightarrow{a \circ b'} & A \longrightarrow 0 \\
& & \beta \downarrow & & \downarrow b' & & \parallel \\
0 & \longrightarrow & K & \xrightarrow{b} & B & \xrightarrow{a} & A \longrightarrow 0
\end{array}$$

such that  $\beta$  is an  $\mathcal{X}$ -epic and  $\alpha$  is a kernel of  $\beta$  by [7, Lemma 2.11]. By [9, Proposition 2.12], the bottom leftmost square is a pullback diagram. Thus  $\beta$  is a deflation by (Ex2). So  $0 \rightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} K$  is in  $\mathcal{L}(\mathcal{A}, \mathcal{X})$ . We are done.  $\square$

*Remark 4.4.* Under the assumption (i) of Proposition 4.3, it can be shown that  $(\mathcal{A}, 0, \mathcal{L}(\mathcal{A}, \mathcal{X}))$  also becomes a partial left triangulated category with  $\mathcal{L}(\mathcal{A}, \mathcal{X})$  the class of all left  $(\mathcal{A}, \mathcal{X})$ -sequences by Proposition 4.1 (i). Dually, under the assumption (ii) of Proposition 4.3,  $(\mathcal{A}, 0, \mathcal{R}(\mathcal{A}, \mathcal{X}))$  also becomes a partial right triangulated category with  $\mathcal{R}(\mathcal{A}, \mathcal{X})$  the class of all right  $(\mathcal{A}, \mathcal{X})$ -sequences by Proposition 4.1 (ii).

**Definition 4.5.** [15, Definition 2.1] Let  $\mathcal{A}$  be an exact category. A *cotorsion pair* in  $\mathcal{A}$  is a pair  $(\mathcal{C}, \mathcal{F})$  of classes of objects of  $\mathcal{A}$  such that

- (a)  $\mathcal{C} = {}^{\perp_1} \mathcal{F} := \{C \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^1(C, F) = 0, \forall F \in \mathcal{F}\}$ .
- (b)  $\mathcal{F} = \mathcal{C}^{\perp_1} := \{F \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^1(C, F) = 0, \forall C \in \mathcal{C}\}$ .

The cotorsion pair  $(\mathcal{C}, \mathcal{F})$  is called *complete* if the following conditions are satisfied:

- (c)  $(\mathcal{C}, \mathcal{F})$  has *enough projectives*, i.e. for each  $A \in \mathcal{A}$  there exists a conflation  $0 \rightarrow F \rightarrow C \rightarrow A \rightarrow 0$  such that  $C \in \mathcal{C}$  and  $F \in \mathcal{F}$ .
- (d)  $(\mathcal{C}, \mathcal{F})$  has *enough injectives*, i.e. for each  $A \in \mathcal{A}$  there exists a conflation  $0 \rightarrow A \rightarrow F \rightarrow C \rightarrow 0$  such that  $C \in \mathcal{C}$  and  $F \in \mathcal{F}$ .

**Proposition 4.6.** *Let  $(\mathcal{C}, \mathcal{F})$  be a complete cotorsion pair in an exact category  $\mathcal{A}$ . Then*

- (i) *there is a class  $\mathcal{L}(\mathcal{A}, \mathcal{C})$  of left  $(\mathcal{A}, \mathcal{C})$ -triangles such that  $(\mathcal{A}, 0, \mathcal{L}(\mathcal{A}, \mathcal{C}))$  is a partial left triangulated category;*
- (ii) *there is a class  $\mathcal{L}(\mathcal{F}, \mathcal{C} \cap \mathcal{F})$  of left  $(\mathcal{F}, \mathcal{C} \cap \mathcal{F})$ -triangles such that  $(\mathcal{F}, 0, \mathcal{L}(\mathcal{F}, \mathcal{C} \cap \mathcal{F}))$  is a partial left triangulated category;*
- (iii) *there is a class  $\mathcal{R}(\mathcal{A}, \mathcal{F})$  of right  $(\mathcal{A}, \mathcal{F})$ -triangles such that  $(\mathcal{A}, 0, \mathcal{R}(\mathcal{A}, \mathcal{F}))$  is a partial right triangulated category;*

(iv) *there is a class  $\mathcal{R}(\mathcal{C}, \mathcal{C} \cap \mathcal{F})$  of right  $(\mathcal{C}, \mathcal{C} \cap \mathcal{F})$ -triangles such that  $(\mathcal{C}, 0, \mathcal{R}(\mathcal{C}, \mathcal{C} \cap \mathcal{F}))$  is a partial right triangulated category.*

*Proof.* (i) By the definition of a complete cotorsion pair, for each object  $A \in \mathcal{F}$  there is a conflation  $0 \rightarrow F \rightarrow C \xrightarrow{p} A \rightarrow 0$  such that  $F \in \mathcal{F}, C \in \mathcal{C}$ . Since  $\mathcal{F} = \mathcal{C}^{\perp_1}$  we know that  $p$  is a  $\mathcal{F}$ -epic. Then the assertion (i) follows from Proposition 4.3 (i) directly. The other statements can be proved similarly.  $\square$

**Example 4.7.** (i) Let  $\mathcal{A}$  be an abelian category with enough projectives. Let  $\mathcal{P}$  be the class of projectives of  $\mathcal{A}$ , then  $(\mathcal{P}, \mathcal{A})$  is a complete cotorsion pair. Thus there is a class  $\mathcal{L}(\mathcal{A}, \mathcal{P})$  of left  $(\mathcal{A}, \mathcal{P})$ -triangles such that  $(\mathcal{A}, 0, \mathcal{L}(\mathcal{A}, \mathcal{P}))$  is a partial left triangulated category. Dually, if  $\mathcal{A}$  has enough injectives and let  $\mathcal{I}$  be the class of injectives, then there is a class  $\mathcal{R}(\mathcal{A}, \mathcal{I})$  of right  $(\mathcal{A}, \mathcal{I})$ -triangles such that  $(\mathcal{A}, 0, \mathcal{R}(\mathcal{A}, \mathcal{I}))$  is a partial right triangulated category.

(ii) Let  $\mathcal{A}$  be an exact category. Recall that a full subcategory  $\mathcal{W}$  of  $\mathcal{A}$  is called *thick* if it is closed under direct summands and if two out of three of the terms in a conflation are in  $\mathcal{W}$ , then so is the third. A triple  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  of classes of objects in an exact category  $\mathcal{A}$  is called a *Hovey triple* if  $\mathcal{W}$  is a thick subcategory of  $\mathcal{A}$  and both  $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are complete cotorsion pairs in  $\mathcal{A}$ . In this case, denote by  $\omega = \mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$ , then there are five partial left triangulated categories  $(\mathcal{W} \cap \mathcal{F}, 0, \mathcal{L}(\mathcal{W} \cap \mathcal{F}, \omega))$ ,  $(\mathcal{A}, 0, \mathcal{L}(\mathcal{A}, \mathcal{C}))$ ,  $(\mathcal{F}, 0, \mathcal{L}(\mathcal{F}, \omega))$ ,  $(\mathcal{A}, 0, \mathcal{L}(\mathcal{A}, \mathcal{C} \cap \mathcal{W}))$  and  $(\mathcal{F}, 0, \mathcal{L}(\mathcal{F}, \mathcal{C} \cap \mathcal{F}))$ . Dually, there are five partial right triangulated categories  $(\mathcal{C} \cap \mathcal{W}, 0, \mathcal{R}(\mathcal{C} \cap \mathcal{W}, \omega))$ ,  $(\mathcal{A}, 0, \mathcal{R}(\mathcal{A}, \mathcal{F}))$ ,  $(\mathcal{C}, 0, \mathcal{R}(\mathcal{C}, \omega))$ ,  $(\mathcal{A}, 0, \mathcal{R}(\mathcal{A}, \mathcal{W} \cap \mathcal{F}))$  and  $(\mathcal{C}, 0, \mathcal{R}(\mathcal{C}, \mathcal{C} \cap \mathcal{F}))$ . In particular, we know that  $\mathcal{F}/\omega$  and  $\mathcal{F}/\mathcal{C} \cap \mathcal{F}$  are left triangulated categories, and  $\mathcal{C}/\omega$  and  $\mathcal{C}/\mathcal{C} \cap \mathcal{F}$  are right triangulated categories [32, Theorem 4.6].

### 4.3. Partial one-sided triangulated categories from one-sided triangulated categories.

Let  $(\mathcal{T}, \Omega, \Delta)$  be a left triangulated category. Then in any left triangle  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  in  $\Delta$ , the morphism  $b$  is a weak kernel of  $a$  and  $k$  is a weak kernel of  $b$  by the dual of Lemma 1.3 of [1]. Let  $\mathcal{X} \subseteq \mathcal{C}$  be two additive subcategories of  $\mathcal{T}$ . Then  $\mathcal{C}$  is said to be *special  $\mathcal{X}$ -epic closed* if the following conditions hold:

(a) For each object  $A \in \mathcal{C}$  there is a left triangle  $\Omega(A) \rightarrow L \rightarrow X \xrightarrow{p} A$  in  $\Delta$  with  $X \in \mathcal{X}$ ,  $L \in \mathcal{C}$  and  $p$  an  $\mathcal{X}$ -epic.

(b) For any morphism  $a: B \rightarrow A$  in  $\mathcal{C}$ , we have  $C \in \mathcal{C}$  in any left triangle  $\Omega(A) \rightarrow C \rightarrow B \oplus X \xrightarrow{(a,p)} A$  in  $\Delta$  with  $X \in \mathcal{X}$  and  $p$  an  $\mathcal{X}$ -epic.

(c) If  $\Omega(A) \xrightarrow{l} L \xrightarrow{x} X \xrightarrow{p} A$  is a left triangle in  $\Delta$  with  $X \in \mathcal{X}$ ,  $L \in \mathcal{C}$  and  $p$  an  $\mathcal{X}$ -epic, then  $x$  is a weak cokernel of  $l$ .

Dually, if  $(\mathcal{T}, \Sigma, \nabla)$  is a right triangulated category and  $\mathcal{X}$  is an additive subcategory of  $\mathcal{T}$ , we can define the notion of a *special  $\mathcal{X}$ -monic closed* additive subcategory of  $\mathcal{T}$ .

**Proposition 4.8.** (i) *Let  $(\mathcal{T}, \Omega, \Delta)$  be a left triangulated category. Let  $\mathcal{X} \subseteq \mathcal{C}$  be two additive subcategories of  $\mathcal{T}$ . If  $\mathcal{C}$  is special  $\mathcal{X}$ -epic closed, then  $(\mathcal{T}, \Omega, \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is a partial left triangulated category with  $\mathcal{L}(\mathcal{C}, \mathcal{X}) = \{\Omega(A) \rightarrow K \rightarrow B \xrightarrow{a} A \in \Delta \mid K \in \mathcal{C}, a \text{ is an } \mathcal{X}\text{-epic in } \mathcal{C}\}$ .*

(ii) *Let  $(\mathcal{T}, \Sigma, \nabla)$  be a right triangulated category. Let  $\mathcal{X} \subseteq \mathcal{C}$  be two additive subcategories of  $\mathcal{T}$ . If  $\mathcal{C}$  is special  $\mathcal{X}$ -monic closed, then  $(\mathcal{T}, \Sigma, \mathcal{R}(\mathcal{C}, \mathcal{X}))$  is a partial right triangulated category with  $\mathcal{R}(\mathcal{C}, \mathcal{X}) = \{A \xrightarrow{b} B \rightarrow K \rightarrow \Sigma(A) \in \nabla \mid K \in \mathcal{C}, b \text{ is an } \mathcal{X}\text{-monic in } \mathcal{C}\}$ .*

*Proof.* (i) We verify (PLT1)-(PLT4) of Definition 2.2 one by one.

(PLT1) The statements (i) and (ii) follow from the fact that both the class of  $\mathcal{X}$ -epics and the class  $\Delta$  are closed under isomorphisms and direct sums.

(PLT2) (i) For each object  $A \in \mathcal{C}$ , since  $\mathcal{C}$  is special  $\mathcal{X}$ -epic closed, there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \rightarrow L \rightarrow X \xrightarrow{p} A$  with  $X \in \mathcal{X}$  and  $L \in \mathcal{C}$ .

(ii) For any morphism  $a: B \rightarrow A$ ,  $\Omega(A) \xrightarrow{0} B \xrightarrow{\begin{pmatrix} -1_B \\ a \end{pmatrix}} B \oplus A \xrightarrow{(a, 1_A)} A$  is in  $\mathcal{L}(\mathcal{C}, \mathcal{X})$  by the dual of [35, Lemma 2.9].

(iii) This follows from the assumption that  $\mathcal{C}$  is special  $\mathcal{X}$ -epic closed.

(PLT3) Assume that we have the following diagram

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A \\ \Omega(\alpha) \downarrow & & \gamma \downarrow & & \downarrow \beta & & \downarrow \alpha \\ \Omega(A') & \xrightarrow{k'} & K' & \xrightarrow{b'} & B' & \xrightarrow{a'} & A \end{array}$$

with rows in  $\mathcal{L}(\mathcal{C}, \mathcal{X})$ . By (LT3), there is a morphism  $\gamma: K \rightarrow K'$  such that the whole diagram commutative. Assume that  $B \in \mathcal{X}$  in the first row of the above diagram and there is a morphism  $\gamma: K \rightarrow K'$  making the above whole diagram commutative. If there is a morphism  $s: A \rightarrow B'$  such that  $\alpha = a' \circ s$ , then  $\gamma \circ k = k' \circ \Omega(\alpha) = k' \circ \Omega(a') \circ \Omega(s) = 0$  by (LT2) and the dual of [1, Lemma 1.3]. Thus there is a morphism  $t: B \rightarrow K'$  such that  $\gamma = t \circ b$  since  $\mathcal{C}$  is special  $\mathcal{X}$ -epic closed and then  $b$  is a weak cokernel of  $k$ .

(PLT4) Let  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$  and  $\Omega(B) \xrightarrow{l} L \xrightarrow{c} C \xrightarrow{b'} B$  be two left  $(\mathcal{C}, \mathcal{X})$ -triangles such that there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \xrightarrow{m} M \xrightarrow{c'} C \xrightarrow{aob'} A$ . By LT(4),

there is commutative diagram

$$\begin{array}{ccccccc}
\Omega(K) & \xlongequal{\quad} & \Omega(K) & & & & \\
\Omega(b) \downarrow & & \downarrow & & & & \\
\Omega(B) & \xrightarrow{l} & L & \xrightarrow{c} & C & \xrightarrow{b'} & B \\
\Omega(a) \downarrow & & \alpha \downarrow & & \parallel & & \downarrow a \\
\Omega(A) & \xrightarrow{m} & M & \xrightarrow{c'} & C & \xrightarrow{a \circ b'} & A \\
\parallel & & \beta \downarrow & & \downarrow b' & & \parallel \\
\Omega(A) & \xrightarrow{k} & K & \xrightarrow{b} & B & \xrightarrow{a} & A
\end{array}$$

such that the second column from the left is a left triangle in  $\Delta$ . We shall show that  $\beta$  is an  $\mathcal{X}$ -epic. For this, let  $s: X \rightarrow K$  be any morphism with  $X \in \mathcal{X}$ . Since  $b'$  is an  $\mathcal{X}$ -epic, there is a morphism  $t: X \rightarrow C$  such that  $b' \circ t = b \circ s$ . Thus  $a \circ b' \circ t = a \circ b \circ s = 0$ . Then there is a morphism  $x: X \rightarrow M$  such that  $c' \circ x = t$ . So  $b \circ s = b' \circ t = b' \circ c' \circ x = b \circ \beta \circ x$ , then there exists a morphism  $x': X \rightarrow \Omega(A)$  such that  $k \circ x' = s - \beta \circ x$ . Then  $s = k \circ x' + \beta \circ x = \beta \circ m \circ x' + \beta \circ x = \beta \circ (m \circ x' + x)$  (here we use the fact that  $\mathcal{C}$  is a full subcategory of  $\mathcal{T}$ ). Therefore  $\beta$  is an  $\mathcal{X}$ -epic. This shows that  $\Omega(K) \xrightarrow{l \circ \Omega(b)} N \xrightarrow{\alpha} M \xrightarrow{\beta} K$  is a left  $(\mathcal{C}, \mathcal{X})$ -triangle.

The statement (ii) can be proved dually.  $\square$

**Corollary 4.9.** *Let  $(\mathcal{T}, [1], \Delta)$  be a triangulated category. Let  $\mathcal{X} \subseteq \mathcal{T}$  be an additive subcategory of  $\mathcal{T}$ .*

(i) *If  $\mathcal{X}$  is contravariantly finite in  $\mathcal{T}$ , then there is a class  $\mathcal{L}(\mathcal{T}, \mathcal{X})$  of left  $(\mathcal{T}, \mathcal{X})$ -triangles such that  $(\mathcal{T}, [-1], \mathcal{L}(\mathcal{T}, \mathcal{X}))$  is a partial left triangulated category.*

(ii) *If  $\mathcal{X}$  is covariantly finite in  $\mathcal{T}$ , then there is a class  $\mathcal{R}(\mathcal{T}, \mathcal{X})$  of right  $(\mathcal{T}, \mathcal{X})$ -triangles such that  $(\mathcal{T}, \mathcal{X}; \mathcal{T}, [1])$  is a partial right triangulated category.*

**Example 4.10.** (i) Let  $(\mathcal{T}, [1], \Delta)$  be a triangulated category. Let  $\mathcal{X}$  and  $\mathcal{C}$  be additive subcategories of  $\mathcal{T}$ . Assume that  $(\mathcal{C}, \mathcal{C})$  forms an  $\mathcal{X}$ -mutation pair in the sense of [24, Definition 2.5]:

(a)  $\mathcal{C}$  is *extension-closed*, i.e., if  $A[-1] \rightarrow C \rightarrow B \rightarrow A$  is a triangle in  $\Delta$  such that  $C, A \in \mathcal{C}$ , then  $B \in \mathcal{C}$ .

(b)  $\mathcal{X} \subseteq \mathcal{C}$  and  $\text{Hom}_{\mathcal{T}}(\mathcal{X}[-1], \mathcal{C}) = 0 = \text{Hom}_{\mathcal{T}}(\mathcal{C}, \mathcal{X}[1])$ .

(c) For any object  $A \in \mathcal{C}$ , there exists triangles  $A[-1] \rightarrow K_A \rightarrow X_A \rightarrow A$  and  $A \rightarrow X^A \rightarrow K^A \rightarrow A[1]$  in  $\Delta$  such that  $X_A, X^A \in \mathcal{X}$  and  $K_A, K^A \in \mathcal{C}$ .

Then  $\mathcal{C}$  is both special  $\mathcal{X}$ -epic closed and special  $\mathcal{X}$ -monic closed [24, Lemma 4.3 (2)]. Thus,  $(\mathcal{T}, [-1], \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is a partial left triangulated category with  $\mathcal{L}(\mathcal{C}, \mathcal{X}) =$

$\{A[-1] \rightarrow K \rightarrow B \xrightarrow{a} A \in \Delta \mid K \in \mathcal{C}, a \text{ is an } \mathcal{X}\text{-epic in } \mathcal{C}\}$  and  $(\mathcal{T}, [1], \mathcal{R}(\mathcal{C}, \mathcal{X}))$  is a partial right triangulated category with  $\mathcal{R}(\mathcal{C}, \mathcal{X}) = \{A \xrightarrow{b} B \rightarrow K \rightarrow A[1] \in \Delta \mid K \in \mathcal{C}, b \text{ is an } \mathcal{X}\text{-monic in } \mathcal{C}\}$ .

(ii) Let  $(\mathcal{T}, \Sigma, \nabla)$  be a right triangulated category such that  $A \xrightarrow{b} B \xrightarrow{c} C \xrightarrow{a} \Sigma(A)$  is a right triangle in  $\nabla$  if  $B \xrightarrow{c} C \xrightarrow{a} \Sigma(A) \xrightarrow{-\Sigma(b)} \Sigma(B)$  is a right triangle in  $\nabla$ . Let  $\mathcal{Y}$  be a *factor-through-epic* additive subcategory of  $\mathcal{T}$  in the sense of [35, Definition 2.7], that is, if a morphism  $f: \Sigma(A) \rightarrow \Sigma(B)$  factors some object in  $\Sigma^n(\mathcal{Y})$  for some positive integer  $n$ , there is  $f': A \rightarrow B$  which factors through some object in  $\Sigma^{n-1}(\mathcal{Y})$  such that  $f = \Sigma(f')$ . Let  $\mathcal{C}$  be an extension-closed additive subcategory of  $\mathcal{T}$  such that  $\mathcal{Y}$  is a covariantly finite subcategory of  $\mathcal{C}$ . If for each object  $A$  in  $\mathcal{C}$  there is a right triangle  $A \xrightarrow{i} Y \rightarrow K \rightarrow \Sigma(A)$  in  $\nabla$  such that  $i$  is a left  $\mathcal{Y}$ -approximation and  $K \in \mathcal{C}$ , then  $(\mathcal{T}, \Sigma, \mathcal{R}(\mathcal{Y}, \mathcal{C}))$  is a partial right triangulated category with  $\mathcal{R}(\mathcal{C}, \mathcal{Y}) = \{A \xrightarrow{b} B \rightarrow K \rightarrow \Sigma(A) \in \nabla \mid K \in \mathcal{C}, b \text{ is a } \mathcal{Y}\text{-monic in } \mathcal{C}\}$ . In fact, in this case, by [35, Lemma 3.3] and the proof of [35, Theorem 3.9],  $\mathcal{C}$  is special  $\mathcal{Y}$ -monic closed.

*Remark 4.11.* By Example 4.10 (ii), we can get [35, Theorem 3.9] from our Theorem 3.2 directly.

Let  $(\mathcal{T}, [1], \Delta)$  is a triangulated category. Recall that a pair  $(\mathcal{X}, \mathcal{Y})$  of additive subcategories of  $\mathcal{T}$  is called a *torsion pair* (i.e. a *torsion theory* in [24, Definition 2.2]) if the following two conditions hold:

- (a)  $\text{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0$ ;
- (b) for any object  $T$  in  $\mathcal{T}$ , there is a triangle  $Y[-1] \rightarrow X \rightarrow T \rightarrow Y$  in  $\Delta$  with  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ .

If  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair in a triangulated category  $\mathcal{T}$ , then  $\mathcal{X}$  is contravariantly finite in  $\mathcal{T}$  and  $\mathcal{Y}$  is covariantly finite in  $\mathcal{T}$ . Thus by Corollary 4.9, we have the following result:

**Corollary 4.12.** *Let  $(\mathcal{X}, \mathcal{Y})$  be a torsion pair in a triangulated category  $(\mathcal{T}, [1], \Delta)$ . Then there is a partial left triangulated category  $(\mathcal{T}, [-1], \mathcal{L}(\mathcal{T}, \mathcal{X}))$  and a partial right triangulated category  $(\mathcal{T}, [1], \mathcal{R}(\mathcal{T}, \mathcal{Y}))$ .*

## 5. STABLE EXACT CATEGORIES

**5.1. Stable preabelian categories.** Recall that an additive category  $\mathcal{C}$  is said to be *preabelian* if every morphism in  $\mathcal{C}$  has a kernel and a cokernel.

The following result is a generalization of [29, Theorem 3.3] (compare [8, Theorem 3.1]):

**Proposition 5.1.** *Let  $(\mathcal{T}, [1], \Delta)$  be a triangulated category and  $\mathcal{X}$  an additive subcategory of  $\mathcal{T}$ . Consider the following conditions:*

- (a) *For each object  $A \in \mathcal{T}$  there is a triangle  $A[-1] \rightarrow X^1 \rightarrow X^0 \xrightarrow{p} A$  in  $\Delta$  such that  $p$  is an  $\mathcal{X}$ -epic and  $X^1, X^0 \in \mathcal{X}$ ;*
- (b) *For each object  $A \in \mathcal{T}$  there is a triangle  $A \xrightarrow{i} X_0 \rightarrow X_1 \rightarrow A[1]$  in  $\Delta$  such that  $i$  is an  $\mathcal{X}$ -monic and  $X_0, X_1 \in \mathcal{X}$ .*
- (c) *In any triangle  $C \xrightarrow{b} B \xrightarrow{a} A \rightarrow C[1]$  in  $\Delta$ ,  $b$  is an  $\mathcal{X}$ -monic if  $\underline{a}$  is an epimorphism in  $\mathcal{T}/\mathcal{X}$  and  $a$  is an  $\mathcal{X}$ -epic if  $\underline{b}$  is a monomorphism*

*If (a) and (b) hold, then  $\mathcal{T}/\mathcal{X}$  is a preabelian category. Furthermore, if (c) holds also, then  $\mathcal{T}/\mathcal{X}$  is an abelian category.*

*Proof.* Assume that the conditions (a) and (b) hold. Then  $\mathcal{X}$  is both contravariantly finite and covariantly finite in  $\mathcal{T}$ , thus  $(\mathcal{T}, [-1], \mathcal{L}(\mathcal{T}, \mathcal{X}))$  is a partial left triangulated category with  $\mathcal{L}(\mathcal{T}, \mathcal{X}) = \{A[-1] \rightarrow K \rightarrow B \xrightarrow{a} A \in \Delta \mid a \text{ is an } \mathcal{X}\text{-epic}\}$  and  $(\mathcal{T}, [1], \mathcal{R}(\mathcal{T}, \mathcal{X}))$  is a partial right triangulated category with  $\mathcal{R}(\mathcal{T}, \mathcal{X}) = \{A \xrightarrow{b} B \rightarrow K \rightarrow A[1] \in \Delta \mid b \text{ is an } \mathcal{X}\text{-monic}\}$  by Corollary 4.9. Then  $(\mathcal{T}/\mathcal{X}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$  is a left triangulated category and  $(\mathcal{T}/\mathcal{X}, \Sigma^{\mathcal{X}}, \nabla^{\mathcal{X}})$  is a right triangulated category by Theorem 3.2. By conditions (a) and (b) and the constructions of  $\Omega_{\mathcal{X}}$  and  $\Sigma^{\mathcal{X}}$ , we know that both  $\Omega_{\mathcal{X}}$  and  $\Sigma^{\mathcal{X}}$  are zero functors. Thus any morphism in  $\mathcal{T}/\mathcal{X}$  has a kernel and a cokernel, i.e.,  $\mathcal{T}/\mathcal{X}$  is a preabelian category.

Now suppose that the condition (c) holds also. Assume that we have an epimorphism  $\underline{a}: \underline{B} \rightarrow \underline{A}$  in  $\mathcal{T}/\mathcal{X}$ . Then there is a left triangle  $0 \rightarrow \underline{C} \xrightarrow{\underline{b}} \underline{B} \xrightarrow{\underline{a}} \underline{A}$  in  $\Delta_{\mathcal{X}}$ . Without loss of generality, we may assume that this triangle is a standard left triangle, i.e. there is a triangle  $A[-1] \rightarrow C \xrightarrow{b} B \xrightarrow{a} A$  in  $\Delta$  such that  $a$  is an  $\mathcal{X}$ -epic. By the rotation axiom of  $\Delta$ ,  $C \xrightarrow{b} B \xrightarrow{a} A \rightarrow C[1]$  is a triangle in  $\Delta$ . By assumption,  $b$  is an  $\mathcal{X}$ -monic, so  $\underline{C} \xrightarrow{\underline{b}} \underline{B} \xrightarrow{\underline{a}} \underline{A} \rightarrow 0$  is a standard right triangle in  $\nabla^{\mathcal{X}}$ . Thus  $\underline{a}$  is a cokernel of  $\underline{b}$ . Dually we can show that any monomorphism is a kernel. So  $\mathcal{T}/\mathcal{X}$  is an abelian category.  $\square$

**Example 5.2.** If  $\mathcal{X}$  is a *tilting subcategory* of  $\mathcal{T}$  in the sense of [29, Definition 3.1] (which is also called a *maximal 1-orthogonal subcategory* in [23]), then  $\mathcal{X}$  satisfies the conditions in (a)-(c) of Proposition 5.1 by [29, Lemma 3.2 and Theorem 2.3]. Thus  $\mathcal{T}/\mathcal{X}$  is an abelian category which is [29, Theorem 3.3].



**5.2. Stable exact categories.** The following proposition improves and generalizes [30, Theorem A (2)] (compare [11, Theorem 3.1] and [12, Theorem 3.5]):

**Proposition 5.3.** *Let  $(\mathcal{A}, \mathcal{E})$  be an exact category and  $\mathcal{X}$  an additive subcategory of  $\mathcal{A}$ . Assume the subcategory  $\mathcal{X}$  satisfies the following conditions:*

- (a) *for each object  $A \in \mathcal{A}$  there is a conflation  $0 \rightarrow X^1 \rightarrow X^0 \xrightarrow{p} A \rightarrow 0$  such that  $p$  is an  $\mathcal{X}$ -epic and  $X^1, X^0 \in \mathcal{X}$ ;*
- (b) *for each object  $A \in \mathcal{A}$ , there is a conflation  $0 \rightarrow A \xrightarrow{i} X_0 \rightarrow X_1 \rightarrow 0$  such that  $i$  is an  $\mathcal{X}$ -monic and  $X_0, X_1 \in \mathcal{X}$ .*

*Then  $\mathcal{A}/\mathcal{X}$  is both a preabelian category and an exact category.*

*Proof.* By Proposition 4.3,  $(\mathcal{A}, 0, \mathcal{L}(\mathcal{A}, \mathcal{X}))$  is a partial left triangulated category with  $L(\mathcal{A}, \mathcal{X}) = \{0 \rightarrow K \rightarrow B \xrightarrow{a} A \mid 0 \rightarrow K \rightarrow B \xrightarrow{a} A \rightarrow 0 \in \mathcal{E}, a \text{ is an } \mathcal{X}\text{-epic}\}$ . Thus  $(\mathcal{A}/\mathcal{X}, \Omega_{\mathcal{X}}, \triangle_{\mathcal{X}})$  is a left triangulated category by Theorem 3.2. By the construction of  $\Omega_{\mathcal{X}}$  and the condition (a), we know that  $\Omega_{\mathcal{X}} = 0$ . Similarly,  $(\mathcal{A}, 0, \mathcal{R}(\mathcal{A}, \mathcal{X}))$  is a partial right triangulated category with  $\mathcal{R}(\mathcal{A}, \mathcal{X}) = \{A \xrightarrow{b} B \rightarrow K \rightarrow 0 \mid 0 \rightarrow A \xrightarrow{b} B \rightarrow K \rightarrow 0, b \text{ is an } \mathcal{X}\text{-monic}\}$  such that  $(\mathcal{A}/\mathcal{X}, \Sigma^{\mathcal{X}}, \nabla^{\mathcal{X}})$  is a right triangulated category with  $\Sigma^{\mathcal{X}} = 0$ . Thus any morphism in  $\mathcal{A}/\mathcal{X}$  has a kernel and a cokernel and then  $\mathcal{A}/\mathcal{X}$  is a preabelian category.

A conflation  $0 \rightarrow C \xrightarrow{b} B \xrightarrow{a} A \rightarrow 0$  in  $\mathcal{E}$  is said to be  $\mathcal{X}$ -complete if  $b$  is an  $\mathcal{X}$ -monic and  $a$  is an  $\mathcal{X}$ -epic. Let  $\mathcal{E}_{\mathcal{X}}$  be the class of kernel-cokernel sequences  $0 \rightarrow \underline{C} \xrightarrow{\underline{b}} \underline{B} \xrightarrow{\underline{a}} \underline{A} \rightarrow 0$  in  $\mathcal{A}/\mathcal{X}$  induced by  $\mathcal{X}$ -complete conflations  $0 \rightarrow C \xrightarrow{b} B \xrightarrow{a} A \rightarrow 0$  in  $\mathcal{A}$ . We will show that  $\mathcal{E}_{\mathcal{X}}$  is an exact structure on  $\mathcal{A}/\mathcal{X}$ .

We first show that if  $0 \rightarrow C \xrightarrow{f} B \xrightarrow{g} A \rightarrow 0$  is  $\mathcal{X}$ -complete, then  $0 \rightarrow \underline{C} \xrightarrow{\underline{b}} \underline{B} \xrightarrow{\underline{a}} \underline{A} \rightarrow 0$  is a kernel-cokernel sequence in  $\mathcal{A}/\mathcal{X}$ . In fact, by the constructions of the standard left triangles and right triangles in  $\mathcal{A}/\mathcal{X}$ , we have a standard left triangle  $0 \rightarrow \underline{C} \xrightarrow{\underline{b}} \underline{B} \xrightarrow{\underline{a}} \underline{A}$  and a standard right triangle  $\underline{C} \xrightarrow{\underline{b}} \underline{B} \xrightarrow{\underline{a}} \underline{A} \rightarrow 0$ . That is to say,  $0 \rightarrow \underline{C} \xrightarrow{\underline{b}} \underline{B} \xrightarrow{\underline{a}} \underline{A} \rightarrow 0$  is a kernel-cokernel sequence in  $\mathcal{A}/\mathcal{X}$ .

Now we prove that  $\mathcal{E}_{\mathcal{X}}$  is an exact structure on  $\mathcal{A}/\mathcal{X}$  by checking axioms of exact categories one by one. We only verify axioms (Ex0), (Ex1) and (Ex2) since the other two can be proved dually.

(Ex0) This is since  $0 \rightarrow X \xrightarrow{1_X} X \rightarrow 0 \rightarrow 0$  is an  $\mathcal{X}$ -complete conflation.

(Ex1) Let  $\underline{b}': \underline{C} \rightarrow \underline{B}$  and  $\underline{a}: \underline{B} \rightarrow \underline{A}$  be two composable deflations in  $\mathcal{A}/\mathcal{X}$ , i.e. there are two kernel-cokernel sequences  $0 \rightarrow \underline{F} \xrightarrow{\underline{c}} \underline{C} \xrightarrow{\underline{b}} \underline{B} \rightarrow 0$  and  $0 \rightarrow \underline{D} \xrightarrow{\underline{b}'} \underline{B} \xrightarrow{\underline{a}} \underline{A} \rightarrow 0$  in  $\mathcal{E}_{\mathcal{X}}$ . Then we have two  $\mathcal{X}$ -complete conflations  $0 \rightarrow F \xrightarrow{c} C \xrightarrow{b'} B \rightarrow 0$  and  $0 \rightarrow D \xrightarrow{b'} B \xrightarrow{a} A \rightarrow 0$  in  $\mathcal{A}$ . By [7, Lemma 2.11], we have a commutative diagram

of conflations in  $\mathcal{A}$ :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{c} & C & \xrightarrow{b'} & B & \longrightarrow & 0 \\
& & \alpha \downarrow & & \parallel & & \downarrow a & & \\
0 & \longrightarrow & E & \xrightarrow{c'} & C & \xrightarrow{a \circ b'} & A & \longrightarrow & 0 \\
& & \beta \downarrow & & \downarrow b' & & \parallel & & \\
0 & \longrightarrow & D & \xrightarrow{b} & B & \xrightarrow{a} & A & \longrightarrow & 0
\end{array}$$

such that  $\beta$  is an  $\mathcal{X}$ -epic with  $\alpha$  as a kernel. It can be verified directly that the bottom leftmost square is a pullback, then  $\beta$  is a deflation and thus  $\beta$  is a cokernel of  $\alpha$ . We claim that the second row is  $\mathcal{X}$ -complete. In fact, since both  $a$  and  $b'$  are  $\mathcal{X}$ -epics we know that  $a \circ b'$  is also an  $\mathcal{X}$ -epic. To see that  $c'$  is an  $\mathcal{X}$ -monic, let  $s: E \rightarrow X$  be any morphism with  $X \in \mathcal{X}$ . Then there is a morphism  $t: C \rightarrow X$  such that  $s \circ \alpha = t \circ c$  since  $c$  is an  $\mathcal{X}$ -monic. Thus  $(s - t \circ c') \circ \alpha = s \circ \alpha - t \circ c' \circ \alpha = s \circ \alpha - t \circ c = 0$  and then there is a morphism  $u: D \rightarrow X$  such that  $s - t \circ c' = u \circ \beta$  since  $\beta$  is a cokernel of  $\alpha$ . Since  $b$  is an  $\mathcal{X}$ -monic, there is a morphism  $v: B \rightarrow X$  such that  $u = v \circ b$ . Therefore  $s = t \circ c' + u \circ \beta = t \circ c' + v \circ b \circ \beta = t \circ c' + v \circ b' \circ c' = (t + v \circ b') \circ c'$ . So  $c'$  is an  $\mathcal{X}$ -monic and then the conflation  $0 \rightarrow E \xrightarrow{c'} C \xrightarrow{a \circ b'} A \rightarrow 0$  is  $\mathcal{X}$ -complete. Thus  $0 \rightarrow \underline{E} \xrightarrow{\underline{c}'} \underline{C} \xrightarrow{\underline{a} \circ \underline{b}'} \underline{A} \rightarrow 0$  is a sequence in  $\mathcal{E}_{\mathcal{X}}$ , in particular,  $\underline{a} \circ \underline{b}'$  is a deflation.

(Ex2) Let  $\underline{a}: \underline{B} \rightarrow \underline{A}$  be a deflation and  $\underline{f}: \underline{A}' \rightarrow \underline{A}$  a morphism in  $\mathcal{A}/\mathcal{X}$ . Without loss of generality, we may assume that  $\underline{f}$  is lifted by  $f: A' \rightarrow A$ . Let  $\underline{d}$  is induced by the  $\mathcal{X}$ -complete conflation  $0 \rightarrow C \xrightarrow{b} B \xrightarrow{a} A \rightarrow 0$  in  $\mathcal{A}$ . Then we have a pullback diagram in  $\mathcal{A}$  by the dual of [9, Proposition 2.12]:

$$\begin{array}{ccccccccc}
(5.4) & & 0 & \longrightarrow & C & \xrightarrow{b'} & B' & \xrightarrow{a'} & A' & \longrightarrow & 0 \\
& & & & \parallel & & \downarrow f' & & \downarrow f & & \\
& & 0 & \longrightarrow & C & \xrightarrow{b} & B & \xrightarrow{a} & A & \longrightarrow & 0
\end{array}$$

In particular, we have a conflation  $0 \rightarrow B' \xrightarrow{\begin{pmatrix} -f' \\ \underline{a}' \end{pmatrix}} B \oplus A' \xrightarrow{(a, f)} A \rightarrow 0$  in  $\mathcal{A}$ . It can be verified directly that both this conflation and the first row of (5.4) are  $\mathcal{X}$ -complete.

So  $\underline{a}'$  is a deflation in  $\mathcal{A}/\mathcal{X}$  and  $0 \rightarrow \underline{B}' \xrightarrow{\begin{pmatrix} -\underline{f}' \\ \underline{a}' \end{pmatrix}} \underline{B} \oplus \underline{A}' \xrightarrow{(a, f)} \underline{A} \rightarrow 0$  is a kernel-cokernel sequence in  $\mathcal{E}_{\mathcal{X}}$ . Thus the pullback diagram (5.4) lifts to a pullback diagram in  $\mathcal{A}/\mathcal{X}$ .  $\square$

**Example 5.5.** Let  $k$  be a field and  $p \geq 2$  a natural number. Let  $\mathbb{X}$  be the weighted projective line of type  $(2, 3, p)$ . Let  $\text{vec-}\mathbb{X}$  be the category of vector bundles and  $\mathcal{F}$  the additive closure of the *fading* line bundles in the sense of [30]. Then  $\text{vec-}\mathbb{X}$  is an exact category and  $\mathcal{F}$  satisfies the conditions of Proposition 5.3 by the proof of

[30, Proposition 4.13]. Then  $\text{vec-}\mathbb{X}/\mathcal{F}$  is both an exact category and a preabelian category. In particular, if  $\text{vec-}\mathbb{X}$  is a Frobenius category, then so is  $\text{vec-}\mathbb{X}/\mathcal{F}$ , this is [30, Theorem A (2)] and [11, Example 3.3].

## 6. STABLE TRIANGULATED CATEGORIES

### 6.1. Partial triangulated categories.

**Definition 6.1.** Let  $\mathcal{A}$  be an additive category with two endofunctors  $\Omega$  and  $\Sigma$ . Assume that  $\mathcal{X} \subseteq \mathcal{C}$  are two additive subcategories of  $\mathcal{A}$ . Let  $\mathcal{L}(\mathcal{C}, \mathcal{X})$  be a class of left  $(\mathcal{C}, \mathcal{X})$ -sequences and  $\mathcal{R}(\mathcal{C}, \mathcal{X})$  a class of right  $(\mathcal{C}, \mathcal{X})$ -sequences. The 5-tuple  $(\mathcal{A}, \Omega, \Sigma, \mathcal{L}(\mathcal{C}, \mathcal{X}), \mathcal{R}(\mathcal{C}, \mathcal{X}))$  is called a *partial triangulated category* if the following conditions hold:

- (a)  $(\mathcal{A}, \Omega, \mathcal{L}(\mathcal{C}, \mathcal{X}))$  is a partial left triangulated category and  $(\mathcal{A}, \Sigma, \mathcal{R}(\mathcal{C}, \mathcal{X}))$  is a partial right triangulated category
- (b)  $(\Omega, \Sigma)$  is an adjoint pair with the adjunction isomorphism  $\psi$ .
- (c) For each object  $A \in \mathcal{C}$ ,  $\Omega(A) \xrightarrow{u} K \xrightarrow{v} X \xrightarrow{p} A$  is a left  $(\mathcal{C}, \mathcal{X})$ -triangle with  $X \in \mathcal{X}$  if and only if  $K \xrightarrow{v} X \xrightarrow{p} A \xrightarrow{-\psi_{A,K}(u)} \Sigma(K)$  is a right  $(\mathcal{C}, \mathcal{X})$ -triangle.

**Theorem 6.2.** Let  $(\mathcal{A}, \Omega, \Sigma, \mathcal{L}(\mathcal{C}, \mathcal{X}), \mathcal{R}(\mathcal{C}, \mathcal{X}))$  be a partial triangulated category. Then  $\mathcal{C}/\mathcal{X}$  is a triangulated category.

*Proof.* By the condition (a) and Theorem 3.2 (i), we know that  $\mathcal{C}/\mathcal{X}$  has a left triangulated structure  $(\Omega_{\mathcal{X}}, \triangle_{\mathcal{X}})$ . So we only need to show that  $\Omega_{\mathcal{X}}$  is an equivalence. This is equivalent to prove that  $\Omega_{\mathcal{X}}$  is dense, full and faithful by Theorem II.2.7 of [14]. For each object  $A$  in  $\mathcal{C}$ , let  $A \xrightarrow{i^A} X^A \xrightarrow{\pi^A} K^A \xrightarrow{\nu^A} \Sigma(A)$  be the chosen right  $(\mathcal{C}, \mathcal{X})$ -triangle of  $A$  with  $X^A \in \mathcal{X}$  and  $\Omega(A) \xrightarrow{\nu^A} K_A \xrightarrow{\iota^A} X_A \xrightarrow{p^A} A$  the chosen left  $(\mathcal{C}, \mathcal{X})$ -triangle of  $A$  with  $X_A \in \mathcal{X}$ .

We first show that  $\Omega_{\mathcal{X}}$  is dense. In fact, given any object  $\underline{A}$  in  $\mathcal{C}/\mathcal{X}$ , by assumption,  $\Omega(K^A) \xrightarrow{-\psi_{K^A,A}^{-1}(\nu^A)} A \xrightarrow{i^A} X^A \xrightarrow{\pi^A} K^A$  is a left  $(\mathcal{C}, \mathcal{X})$ -triangle with  $X^A \in \mathcal{X}$ . By the construction of  $\Omega_{\mathcal{X}}$  and Lemma 2.4 (ii),  $\underline{A} \cong \Omega_{\mathcal{X}}(\underline{K^A})$ , so  $\Omega_{\mathcal{X}}$  is dense.

For the fullness of  $\Omega_{\mathcal{X}}$ , let  $\underline{A}, \underline{B}$  be two objects in  $\mathcal{C}/\mathcal{X}$  and  $\underline{f}: \Omega_{\mathcal{X}}(\underline{A}) \rightarrow \Omega_{\mathcal{X}}(\underline{B})$  a morphism in  $\mathcal{C}/\mathcal{X}$ . By the construction of  $\Omega_{\mathcal{X}}$ ,  $\Omega_{\mathcal{X}}(\underline{A}) = \underline{K_A}$  and  $\Omega_{\mathcal{X}}(\underline{B}) = \underline{K_B}$ . Consider the following commutative diagram of right  $(\mathcal{C}, \mathcal{X})$ -triangles

$$\begin{array}{ccccccc}
 K_A & \xrightarrow{\iota_A} & X_A & \xrightarrow{p_A} & A & \xrightarrow{-\psi_{A,K_A}(\nu_A)} & \Sigma(K_A) \\
 f \downarrow & & x \downarrow & & y \downarrow & & \downarrow \Sigma(f) \\
 K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{p_B} & B & \xrightarrow{-\psi_{B,K_B}(\nu_B)} & \Sigma(K_B)
 \end{array}$$

where the existence of  $x$  is since  $\iota_A$  is an  $\mathcal{X}$ -monic by assumption (c), and the existence of  $y$  is by the axiom (PRT3) of a partial right triangulated category. We claim that  $\Omega_{\mathcal{X}}(\underline{y}) = \underline{f}$ . In fact, follow from the naturality of  $\psi_{A,K_A}$  in  $K_A$ , we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(\Omega(A), K_A) & \xrightarrow{\psi_{A,K_A}} & \mathrm{Hom}_{\mathcal{A}}(A, \Sigma(K_A)) \\ \mathrm{Hom}_{\mathcal{A}}(\Sigma(A), f) \downarrow & & \downarrow \mathrm{Hom}_{\mathcal{A}}(A, \Sigma(f)) \\ \mathrm{Hom}_{\mathcal{A}}(\Omega(A), K_B) & \xrightarrow{\psi_{A,K_B}} & \mathrm{Hom}_{\mathcal{A}}(A, \Sigma(K_B)) \end{array}$$

and then for  $\nu_A \in \mathrm{Hom}_{\mathcal{A}}(\Omega(A), K_A)$  we get

$$\psi_{A,K_B}(f \circ \nu_A) = \Sigma(f) \circ \psi_{A,K_A}(\nu_A).$$

Similarly, by the naturality of  $\psi_{A,K_B}$  in  $A$ , we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(\Omega(B), K_B) & \xrightarrow{\psi_{B,K_B}} & \mathrm{Hom}_{\mathcal{A}}(B, \Sigma(K_B)) \\ \mathrm{Hom}_{\mathcal{A}}(\Sigma(y), K_B) \downarrow & & \downarrow \mathrm{Hom}_{\mathcal{A}}(y, \Sigma(K_B)) \\ \mathrm{Hom}_{\mathcal{A}}(\Omega(A), K_B) & \xrightarrow{\psi_{A,K_B}} & \mathrm{Hom}_{\mathcal{A}}(A, \Sigma(K_B)) \end{array}$$

This yields for  $\nu_B \in \mathrm{Hom}_{\mathcal{C}}(\Omega(B), K_B)$  the formula

$$\psi_{A,K_B}(\nu_B \circ \Omega(y)) = \psi_{B,K_B}(\nu_B) \circ y.$$

Since  $\Sigma(f) \circ \psi_{A,K_A}(\nu_A) = \psi_{B,K_B}(\nu_B) \circ y$  by the construction of  $y$  and the fact that  $\psi_{A,K_B}$  is an isomorphism, we have

$$\nu_B \circ \Omega(y) = f \circ \nu_A.$$

So we have the following commutative diagram of left  $(\mathcal{C}, \mathcal{X})$ -triangles:

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{\nu_A} & K_A & \xrightarrow{\iota_A} & X_A & \xrightarrow{p_A} & A \\ \Omega(y) \downarrow & & \downarrow f & & x \downarrow & & \downarrow y \\ \Omega(B) & \xrightarrow{\nu_B} & K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{p_B} & B \end{array}$$

This shows that  $\Omega_{\mathcal{X}}(\underline{y}) = \underline{f}$ . Thus  $\Omega_{\mathcal{X}}$  is full.

To see that  $\Omega_{\mathcal{X}}$  is faithful, take a morphism  $\underline{g} \in \mathrm{Hom}_{\mathcal{C}/\mathcal{X}}(\underline{A}, \underline{B})$ . By the construction of  $\Omega_{\mathcal{X}}(\underline{g})$ , we have the following commutative diagram

$$\begin{array}{ccccccc} \Omega(A) & \xrightarrow{\nu_A} & K_A & \xrightarrow{\iota_A} & X_A & \xrightarrow{p_A} & A \\ \Omega(g) \downarrow & & \downarrow \kappa_g & & x_g \downarrow & & \downarrow g \\ \Omega(B) & \xrightarrow{\nu_B} & K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{p_B} & B \end{array}$$

with  $\Omega_{\mathcal{X}}(\underline{g}) = \underline{\kappa}_g$ . Applying the naturality of  $\psi_{A,K_A}$  in  $A$  and  $K_A$  again, we can prove that the above commutative diagram induces the following commutative diagram of right  $(\mathcal{X}, \mathcal{C})$ -triangles:

$$\begin{array}{ccccccc} K_A & \xrightarrow{\iota_A} & X_A & \xrightarrow{p_A} & A & \xrightarrow{-\psi_{A,K_A}(\nu_A)} & \Sigma(K_A) \\ \kappa_g \downarrow & & x_g \downarrow & & g \downarrow & & \downarrow \Sigma(\kappa_g) \\ K_B & \xrightarrow{\iota_B} & X_B & \xrightarrow{p_B} & B & \xrightarrow{-\psi_{B,K_B}(\nu_B)} & \Sigma(K_B) \end{array}$$

If  $\underline{\kappa}_g = 0$ , i.e.,  $\kappa_g$  factors through some object in  $\mathcal{X}$ , then it factors through  $\iota_A$  since  $\iota_A$  is an  $\mathcal{X}$ -monic by assumption (c). Thus  $g$  factors through  $p_B$  by the dual of (PLT3). So,  $\underline{g} = 0$  in  $\mathcal{C}/\mathcal{X}$  and then  $\Omega_{\mathcal{X}}$  is faithful.  $\square$

**Corollary 6.3.** ([24, Theorem 4.2]) *Let  $(\mathcal{T}, [1])$  be a triangulated category. Let  $\mathcal{X} \subseteq \mathcal{C}$  be two additive subcategories of  $\mathcal{T}$ . If  $(\mathcal{C}, \mathcal{C})$  forms an  $\mathcal{X}$ -mutation pair, then  $\mathcal{C}/\mathcal{X}$  is a triangulated category.*

*Proof.* By Example 4.10 (i), there are classes  $\mathcal{L}(\mathcal{C}, \mathcal{X})$  and  $\mathcal{R}(\mathcal{C}, \mathcal{X})$  of left and right  $(\mathcal{C}, \mathcal{X})$ -sequences such that  $(\mathcal{T}, [-1], [1], \mathcal{L}(\mathcal{C}, \mathcal{X}), \mathcal{R}(\mathcal{C}, \mathcal{X}))$  is a partial triangulated category. Also note that the triangulated structure  $(\Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$  coincides with the one in [24], so the claim follows from Theorem 6.2.  $\square$

**Corollary 6.4.** ([17, Theorem 2.6]) *Let  $\mathcal{F}$  be a Frobenius category. Let  $\mathcal{I}$  be the subcategory of projective-injective objects of  $\mathcal{F}$ . Then the stable category  $\mathcal{F}/\mathcal{I}$  is a triangulated category.*

*Proof.* By Example 4.7 (i), there are classes  $\mathcal{L}(\mathcal{F}, \mathcal{I})$  and  $\mathcal{R}(\mathcal{F}, \mathcal{I})$  of left and right  $(\mathcal{F}, \mathcal{I})$ -sequences such that  $(\mathcal{F}, 0, 0, \mathcal{L}(\mathcal{F}, \mathcal{I}), \mathcal{R}(\mathcal{F}, \mathcal{I}))$  is a partial triangulated category by. So  $\mathcal{F}/\mathcal{I}$  is a triangulated category by Theorem 6.2.  $\square$

*Remark 6.5.* Our Theorem 6.2 also covers [38, Theorem 6.17] by noting the proof of [38, Proposition 6.9] and [38, Condition 6.1].

**6.2. A Quillen closed model structure of Iyama-Yoshino subfactor triangulated categories.** Let  $\mathcal{C}$  be an additive category. Recall that a *closed model structure* in the sense of Quillen [36, Definition I.5.1] on  $\mathcal{C}$  consists of three classes of morphisms called *cofibrations*, *fibrations* and *weak equivalences*, denoted by  $\mathcal{Cof}(\mathcal{C})$ ,  $\mathcal{Fib}(\mathcal{C})$  and  $\mathcal{We}(\mathcal{C})$  respectively, which satisfy some axioms. For details, see [4, Definition 4.1]. For standard material of model categories, we refer the reader to [36, Chapter I], [13], [21, Chapter 1] and [22, Chapter 8].

If idempotents split in  $\mathcal{C}$ , for any additive subcategory  $\mathcal{X}$  of  $\mathcal{C}$ , define classes of morphisms in  $\mathcal{C}$  as follows: (i)  $\mathcal{Cof}_{\mathcal{X}}(\mathcal{C})$  is the class of  $\mathcal{X}$ -monics; (ii)  $\mathcal{Fib}_{\mathcal{X}}(\mathcal{C})$  is

the class of  $\mathcal{X}$ -epics; (iii)  $\mathcal{W}e(\mathcal{C})$  is the class of stable equivalences. Recall that a morphism  $f: B \rightarrow A$  in  $\mathcal{C}$  is called a *stable equivalence* if there is a morphism  $f': A \rightarrow B$  such that both  $f \circ f' - 1_A$  and  $f' \circ f - 1_B$  factor through  $\mathcal{X}$ . If  $\mathcal{X}$  is functorially finite in  $\mathcal{C}$  (that is,  $\mathcal{X}$  is both contravariantly and covariantly finite in  $\mathcal{C}$ ), the triple  $(\mathcal{C}of_{\mathcal{X}}(\mathcal{C}), \mathcal{F}ib_{\mathcal{X}}(\mathcal{C}), \mathcal{W}e_{\mathcal{X}}(\mathcal{C}))$  is a closed model structure on  $\mathcal{C}$  by [4, Theorem 4.5], denoted by  $\mathcal{M}_{\mathcal{X}}$ , and the associated homotopy category  $\mathbf{Ho}(\mathcal{M}_{\mathcal{X}})$  is equivalent to the stable category  $\mathcal{C}/\mathcal{X}$ .

The following improves [4, Theorem 4.5]:

**Theorem 6.6.** *Let  $(\mathcal{A}, \Omega, \Sigma, \mathcal{L}(\mathcal{C}, \mathcal{X}), \mathcal{R}(\mathcal{C}, \mathcal{X}))$  be a partial triangulated category. If idempotents split in  $\mathcal{C}$  and every  $\mathcal{X}$ -epic can be embedded in a left  $(\mathcal{C}, \mathcal{X})$ -triangle, then the closed model structure  $\mathcal{M}_{\mathcal{X}}$  induces a triangulated structure on  $\mathcal{C}/\mathcal{X}$  which coincides with the triangulated structure  $(\Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ .*

*Proof.* By the construction of  $\mathcal{M}_{\mathcal{X}}$ , every object  $A$  in  $\mathcal{C}$  is bifibrant and  $A \oplus X_A$  is a very good path object for  $A$ :

$$A \xrightarrow{\begin{pmatrix} 1_A \\ 0 \end{pmatrix}} A \oplus X_A \xrightarrow{\begin{pmatrix} 1_A & p_A \\ 1_A & 0 \end{pmatrix}} A \oplus A$$

where  $p_A: X_A \rightarrow A$  is in the chosen left  $(\mathcal{C}, \mathcal{X})$ -triangle for  $A$  with  $X_A \in \mathcal{X}$ . For each morphism  $a: B \rightarrow A$ , there is a commutative diagram of left  $(\mathcal{X}, \mathcal{C})$ -triangles:

$$\begin{array}{ccccccc} \Omega(B) \oplus \Omega(B) & \xrightarrow{(0, \nu_B)} & K_B & \xrightarrow{\begin{pmatrix} 0 \\ \iota_B \end{pmatrix}} & B \oplus X_B & \xrightarrow{\begin{pmatrix} 1_B & p_B \\ 1_B & 0 \end{pmatrix}} & B \oplus B \\ \begin{pmatrix} \Omega(a) & 0 \\ 0 & \Omega(a) \end{pmatrix} \downarrow & & \kappa_a \downarrow & & \begin{pmatrix} a & 0 \\ 0 & x_a \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \\ \Omega(A) \oplus \Omega(A) & \xrightarrow{(0, \nu_A)} & K_A & \xrightarrow{\begin{pmatrix} 0 \\ \iota_A \end{pmatrix}} & A \oplus X_A & \xrightarrow{\begin{pmatrix} 1_A & p_A \\ 1_A & 0 \end{pmatrix}} & A \oplus A \end{array}$$

By [4, Theorem 4.5], the left or right homotopy relation induced from the closed model structure  $\mathcal{M}_{\mathcal{X}}$  coincides with the stable equivalence relation. Thus we can define an endofunctor on  $\mathcal{C}/\mathcal{X}$  by sending  $A$  to  $\underline{K}_A$  and  $a$  to  $\underline{\kappa}_a$  along Quillen's construction as in [36, Section I.2, Page 2.9, Theorem 2]. This is just the functor  $\Omega_{\mathcal{X}}$  on  $\mathcal{C}/\mathcal{X}$  constructed in Subsection 2.2, so it is well-defined.

Given any fibration  $a: B \rightarrow A$  in  $\mathcal{C}$ , by assumption, there is a left  $(\mathcal{C}, \mathcal{X})$ -triangle  $\Omega(A) \xrightarrow{k} K \xrightarrow{b} B \xrightarrow{a} A$ . Since  $\mathcal{C}$  is an additive category,  $\Omega_{\mathcal{X}}(\underline{A})$  is a group object in  $\mathcal{C}/\mathcal{X}$  and giving  $\underline{K}$  a group action of  $\Omega_{\mathcal{X}}(\underline{A})$  is equivalent to giving a morphism from  $\Omega_{\mathcal{X}}(\underline{A})$  to  $\underline{K}$ . See [3, Subsection 1.1]. Thus we can define a left triangle associated with the fibration  $a: B \rightarrow A$  to be

$$\Omega_{\mathcal{X}}(\underline{A}) \rightarrow \underline{K} \rightarrow \underline{B} \xrightarrow{a} \underline{A}$$

which is just the standard left triangle of  $a$ . By Theorem 6.2, the class of the standard left triangles and the endofunctor  $\Omega_{\mathcal{X}}$  induces a triangle structure on  $\mathcal{C}/\mathcal{X}$ .  $\square$

**Corollary 6.7.** *Let  $(\mathcal{T}, [1])$  be a triangulated category. Let  $\mathcal{X}, \mathcal{C}$  be additive subcategories of  $\mathcal{T}$ . If  $(\mathcal{C}, \mathcal{C})$  forms an  $\mathcal{X}$ -mutation pair, then  $\mathcal{M}_{\mathcal{X}}$  is a closed model structure on  $\mathcal{C}$  and the associated homotopy category is equivalent to Iyama-Yoshino subfactor category  $\mathcal{C}/\mathcal{X}$  as triangulated categories.*

*Proof.* Since  $\mathcal{T}$  is a triangulated category and  $\mathcal{C}$  is closed under direct summands, we know that idempotents split in  $\mathcal{C}$ . Thus  $(\text{Cof}_{\mathcal{X}}(\mathcal{C}), \text{Fib}_{\mathcal{X}}(\mathcal{C}), \text{We}_{\mathcal{X}}(\mathcal{C}))$  is a closed model structure on  $\mathcal{C}$ . Then the claim follows from Theorem 6.6 and Corollary 6.3.  $\square$

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